RANKS, SUBDEGREES AND SUBORBITAL GRAPHS OF FINITE PERMUTATION GROUPS

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DECLARATION AND RECOMMENDATION

DECLARATION

This thesis is my original work and has not been submitted to any university for any award.

Signature…………………………………………. Date………………………………

Gachogu Rose Wambui

SD121/0486/14

RECOMMENDATION

This thesis has been submitted for examination with our approval as university supervisors according to Egerton University regulations.

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DEDICATION

I dedicate this job to my beloved family: My husband Francis and our children, Beatrice, Teresa, Jane and Martin.

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ABSTRACT

Recent research has seen the emergence of some algebraic structures through blending of group theory, combinatorics and graph theory. The structure (G, X) is a transitive group G acting on a set X. The concepts of rank, subdegrees and suborbital graphs of (G, X) have formed a subject of recent study through variations of G and X. Several studies have taken into account the action of various subgroups of the modular group on the set of rationals including infinity ($\hat{\mathbb{Q}}$). Recently the action of the symmetric group S_n on various sets has been thoroughly worked on in relation to ranks, subdegrees and suborbital graphs. However, not much has been done on the action of the subgroups of S_n. In view of this, the study focused on the action of the dihedral group (D_n) and the cyclic group $C_n=\langle(12\ldots n)\rangle$ on unordered and ordered r-element subsets $X=\{1, 2, \ldots, n\}$. Each of the actions on unordered subsets has been proved transitive, if and only if $r=1$, $r=n-1$ or $r=n$. This was determined by using the *orbit-stabilizer theorem* and *Cauchy-Frobenius lemma* on each action, G on X, under consideration. The rank of C_n on X was shown to be n, while that of D_n on X was $(n+1)/2$, when n is even and $(n+2)/2$ when n is odd. This was acheived by applying *Cauchy-Frobenius lemma* on the action of the stabilizer of x on X to count the number of orbits of X under the action of the stabilizer. The subdegrees were then deduced by counting the elements of each suborbit, by analyzing the action of the stabilizer on X. *Sim's theory* was then employed to construct suborbital graphs corresponding to the actions. The construction realized 3 graphs whose properties have been discussed. The study also examined the action of a cyclic subgroup of the projective special linear group on finite subsets of the set of integers, $\mathbb{Z}_p$ (integers modulo p), where the action was proved transitive, rank was shown to be p and 1 connected graph was constructed. The ranks and subdegrees are significant in determination of distance-transitive representations of the linear groups and also in characterization of rank 3 permutation groups. Some group-theoretical properties are also studied through suborbital graphs. The choice of finite sets will familiarize aspiring researchers in the subject. The results have been used to investigate primitivity of the groups, which offers an opening for further research. It is expected that the results will also provide a tool for studies in Category theory, Structure and bonding in Chemistry, Hadamard matrices and Data structures in Computer science.

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LIST OF SYMBOLS

Γ_i	The suborbital graph corresponding to the suborbital O_i
$\lvert Fix(g)\rvert$	Number of elements in X fixed by $g \in G$
Δ	Suborbit of G on X
Δ^*	The G-suborbit paired with Δ
$\boldsymbol{O}$	The suborbital of G on $X \times X$
C_n	Cyclic group of order n generated by the element $(12\ldots n)$
D_n	Dihedral group of degree n and order $2n$
$PGL(n,q)$	The projective general linear group
$PSL(n,q)$	The projective special linear group
S_n	Symmetric group of degree n and order $n!$
$Stab_G(x)$	Stabilizer of an element x in X
$X^{(\boldsymbol{r})}$	Set of all unordered r-element subsets of $X = \{1, 2, \ldots, n\}$
$X^{[\boldsymbol{r}]}$	Set of all ordered r-element subsets of $X = \{1,2, \ldots, n\}$
$GF(q)$	Galois field of q elements, $q = p^{\propto}$, for p a prime and $\propto$ a positive integer
$\mathbb{Q}$	The set of all rationals
$\mathbb{Z}$	The set of all integers
$\widehat{\mathbb{Q}}$	The rational projective line, $\mathbb{Q} \cup \{\infty\}$
$PSL(2,\mathbb{Z})$	The modular group, $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1,\ a, b, c, d \in \mathbb{Z}\}$
$\Gamma_0(N)$	The congruence subgroup of the modular group, $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \ (mod\ n)\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$
$\binom{n}{r}$	All combinations of r from n

CHAPTER ONE

INTRODUCTION

1.1 Background to the study

The subject on ranks and subdegrees is based on the idea of a group action on a set. A group action on a set is a process of developing an algebraic structure through a relation defined between the permutation group G and a set X. The process suppresses major dependence on the group concept, emphasizing the permutation aspect and generalizing the pair (G, X) to have a wider application among other algebras. Such algebraic structures not only reveal connection between different areas in mathematics but also make use of known results in one area to suggest conjectures in a related area. Techniques from one area can also be used to prove results in a related area.

The study, connects three areas of mathematics. Namely; group theory, graph theory and combinatorics. The group acts by permuting the elements of X, from which transitivity is first determined. The concepts of rank and subdegrees have been studied through several permutation groups in which some group theoretic properties have been studied through graphical properties. The area of combinatorics has been used to compute the rank and subdegrees of the transitive actions.

The group action of G on X is a relation on the pair (G, X) where $gx \in X$ is a unique image of every $x \in X$ and $g \in G$ such that

- $Ix = x$ for all $x \in X$, and I is the identity element in G.
- $g_1(g_2x)=(g_1g_2)x$ for all $g \in G$ and $x \in X$.

If G acts on X, then X is partitioned into disjoint equivalence classes called orbits. For each x in X, the orbit of x is the subset of X, $orb_G(x)=\{gx \mid g \in G\}$. The action of G on X is said to be transitive if for every pair $x_i, x_j \in X$, there exists $g \in G$, such that $gx_i= x_j$. Thus, the action has only one orbit. In this case the action is termed as simply transitive. The action is doubly transitive or 2-fold transitive if for any two ordered pairs of distinct elements (x_1, x_2) and (y_1, y_2) in X there exists $g \in G$ such that $y_1=gx_1$ and $y_2=gx_2$. Similarly, k-fold transitivity could be defined. The study will use techniques for actions which are simply transitive (Neumann, 1977).

Previous studies on ranks, subdegrees and suborbital graphs have focused on the action of the subgroups of the modular group on subsets of the rational projective line (Kamuti *et al*., 2012) due to their significance in the arithmetic of elliptic curves, integral quadratic forms and modular forms (Schoeneberg, 1974; Kulkami, 1991). A lot has also been done on the action of the symmetric group S_n on ordered and unordered subsets of $X=\{1, 2, \ldots, n\}$ by several authors. Knowledge of ranks and subdegrees is significant in identification of rank 3 graphs (Habaut, 1975) and determination of existence of distance-transitive graphs (Ivanov, 1984).

1.2 Statement of the problem

The study aims at computing the ranks, subdegrees and suborbital graphs of the groups, D_n and C_n, acting on on $X^{(r)}$ and $X^{[r]}$ among other investigations. The study investigates the properties of suborbits and suborbital graphs of the groups acting on specified sets as follows

i) D_n acting on unordered and ordered r-element subsets of $X=\{1, 2, \ldots, n\}$

ii) $C_n=\langle (12\ldots n) \rangle$ acting on unordered and ordered r-element subsets of $X=\{1, 2, \ldots, n\}$

iii) $H = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$, a subgroup of $PSL(2, q)$, acting on the finite field $\mathbb{Z}_p$

1.3 Objectives

1.3.1 General Objective

To investigate the properties of suborbital graphs, through ranks and subdegrees, associated with the actions of permutation groups on finite sets.

1.3.2 Specific Objectives

i) To determine whether the action is transitive

ii) To compute ranks, suborbits and subdegrees of the action

iii) To construct suborbital graphs of the action and discuss their properties

1.4 Assumptions of the study

i) For every $x \in X$, gx is defined in X.

ii) The actions are not all doubly transitive

iii) The set $Stab_G(x)$ is not a maximal improper subgroup of G in all the actions

1.5 Justification

The action of the groups on finite sets uses an enjoyable and comprehensive approach from specific to general cases. The graphs are useful in the study of Hadamard matrices whose application in computers is essential, especially in error-correcting codes. It will also familiarize aspiring researchers with recent areas in algebra. When group theoretical properties are reflected graphically, the artistic value of mathematics will be portrayed.

1.6 Definition of terms

Definition 1.6.1

A group G, is cyclic if every element of G is a power of a fixed element, g $\in G$. The group is said to be generated by the element g, denoted by $G=< g >$.

The dihedral group, D_n, is the group of all symmetries of a regular polygon with n sides. The group is of order $2n$ and it is generated by a rotation of order n and a reflection of order 2.

Thus, $D_n =\{< x, y : x^n = y^2 = 1 >\}$, where x is a rotation and y a reflection.

Definition 1.6.2 (The Galois Field, $GF(q)$)

A field is a set with two binary operations of addition and multiplication in which the non-zero elements form a group under multiplication. The Galois field with q elements, $GF(q)$, is a finite field where q is a power of a prime.

Definition 1.6.3 (General Linear Group)

The general linear group $GL(n, q)$ is the multiplicative group of all n x n invertible matrices with entries from a field with q elements. The special linear group $SL(n, q)$ is the group of all n x n invertible matrices over $GF(q)$ with determinant 1. The projective special linear group $PSL(n, q)$ is the quotient of $SL(n, q)$ by its centre.

Definition 1.6.4 (Projective General Linear)

The group $PGL(2, q)$ over the Galois field $GF(q)$, is the group consisting of all linear fractional transformations of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in GF(q)$ such that $ad - bc \neq 0$. The elements of the group act on $x \in GF(q)$ via the transform; $x \to \frac{ax+b}{cx+d}$.

Definition 1.6.5 (The modular group, Γ)

The modular group is the projective special linear group $PSL(2, \mathbf{Z})$ with integer entries and determinant 1. The modular group has a subgroup $\Gamma_0(N)$, the congruence subgroup, whose elements are of the form; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \equiv 0 \bmod N$.

Definition 1.6.6 (Fixed point set)

Let G be a group acting on a finite set X. The set of all elements $x \in X$ fixed by $g \in G$ is known as the fixed point set of g, denoted by, $Fix\ (g)=\{x\in X: gx=x\}$.

Definition 1.6.7 (Stabilizer)

Let G act on a set X. The stabilizer of a point $x \in X$ is the set of all elements $g \in G$ which fix x, denoted by $G_x=\{ g \in G \mid gx=x \}$. The set is also denoted by $stab_G(x)$.

Definition 1.6.8 (Rank)

Let G act transitively on a set X. The orbits, $\Delta_0=\{x\}$, Δ_1, …, Δ_{r-1} of G_x on X are known as suborbits of G. The rank of G, in this case, is r and the sizes $|\Delta_i|$ (i=0, 1, …, r-1) the lengths of the suborbits are known as the subdegrees of G.

Definition 1.6.9 (Paired suborbits)

Let G act transitively on a set X and let Δ be an orbit of G_x on X. Define $\Delta^*=\{gx|\ g\in G, x\in g\Delta\}$. Then Δ^* is also an orbit of G_x and is called the G_x-orbit paired with Δ. If $\Delta=\Delta^*$, then Δ is said to be self-paired.

Definition 1.6.10 (A Graph)

A graph is a diagram consisting of an ordered pair (V, E), where V is a finite set whose members are called vertices and E a set of 2-element subsets of V called edges. If $(u ,v) \in E$, then the vertices u and v are said to be adjacent. The graph is a plane figure resulting from joining u and v with a line or a curve, whenever u and v are adjacent. The graph is undirected if for every $(u, v) \in E$, $(v, u) \in E$. The graph is directed if for every $(u, v) \in E$, $(v, u) \notin E$.

Definition 1.6.11 (A path)

For any graph $\mathcal{H}$, a walk is a finite sequence of edges of the form, $v_0v_1, v_1v_2, \dots, v_{m-1}v_m$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 0, 1, \dots, m$-1. The integer m is called the length of the walk.

In terms of vertices, a walk is a sequence of adjacent vertices. If $v_0=v_m$, then the walk is a closed one. A walk may have the same vertex appearing more than one time in the sequence and may not end where it started. A walk in which no vertex appears more than once is a path.

Definition 1.6.12 (Girth)

If every pair of vertices is connected by a path in $\mathcal{H}$, then the graph is said to be connected. In such a case, the closed walk is called a cycle or a circuit. The length of the shortest cycle of $\mathcal{H}$ is called its girth. If $\mathcal{H}$ is connected with no cycles, then it is a tree. The graph $\mathcal{H}$ is a forest if it is a union of trees.

Definition 1.6.13 (Connected component)

A connected component is a maximal connected subgraph of a graph.

Definition1.6.14 (Suborbital graph)

Suppose G is a transitive group acting on X. The action of G on $X \times X$ is defined by; $g(x, y)=(gx, gy)$, $g \in G$, $x, y \in X$. The orbits of this action are known as suborbitals of G. The orbit containing (x, y) is denoted by $O(x, y)$. Let $O_i \subseteq X \times X$, $i=0, 1, \ldots, m-1$ be suborbitals of G. The suborbital graph Γ_i corresponding to the suborbital O_i is formed by taking elements of X as the set of vertices of Γ_i and by drawing a directed edge from x to y if and only if $(x, y) \in O_i$. Hence each suborbital O_i determines a suborbital graph Γ_i.

If $O \subseteq X \times X$ is a G-orbit, then for a fixed $x \in X$, $\Delta=\{y \in X|(x, y) \in O\}$is a G_x-orbit. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $O=(gx, gy)| g \in G, y \in \Delta\}$ is a G-orbit on $X \times X$. In this case, Δ is said to correspond to O. If $x=y$, then $O(x, x)$ is the diagonal of $X \times X$ and the corresponding graph is the trivial suborbital graph which consists of a loop based at each vertex $x \in X$.

Definition 1.6.15 (undirected graph)

Let Δ_i correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired.

Theorem 1.6.16

Let G be transitive on X. Then G is primitive if and only if each non-trivial suborbital graph is connected (Sims, 1967).

Theorem 1.6.17

Let G act transitively on a set X, and suppose $g \in G$. The number of self-paired suborbits of G is given by $\frac{1}{|G|}\sum_{g\in G} |Fix(g^2)|$ (Cameron, 1994).

Theorem 1.6.18

Let G act transitively on X. Then G_x has an orbit different from $\{x\}$ and paired with itself if and only if G is of even order (Wielandt, 1964).

Theorem1.6.19 (The Orbit-stabilizer theorem)

Let G be a group acting on a finite set X and $orb_G(x)$ be the orbit of $x\in X$. The size of $orb_G(x)$ is the index $|G: Stab_G(x)|$ (Rose, 1978, p.72).

Theorem 1.6.20 (Cauchy - Frobenius Lemma)

Let G be a group acting on a finite set X. The number of G- orbits on X is given by

$\frac{1}{|G|}\sum_{g\in G} |Fix(g)|$, (Harary, 1969).

Definition 1.6.21 (Equivalent actions)

Let (G_1, S_1) and (G_2, S_2) be permutation groups (i.e. G_i acts on S_i). The permutation isomorphism, $(G_1, S_1) \equiv (G_2, S_2)$, means there exists a group isomorphism $\phi : G_1 \rightarrow G_2$ and a bijection $\Theta : S_1 \rightarrow S_2$ so that $\Theta(xs)=\phi x(\Theta s)$ for all $x \in G_1$, $s \in S_1$.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter reviews the various work that has been done on ranks, subdegrees and suborbital graphs. Section 2.2 reviews what has been done on ranks and subdegrees, Section 2.3 on suborbital graphs and Section 2.4 on transitivity and primitivity. Section 2.5 gives a brief summary to reveal the gap which the study focuses on.

2.2 Ranks and subdegrees

The rank of the symmetric group S_n acting on 2-elements subsets of the set $X = \{1, 2, \dots, n\}$ was shown to be 3 and the subdegrees as; $1,\ 2(n-2),\ \binom{n-2}{2},$ for $n \geq 4$ (Higman, 1970).

The study on S_n acting on 2-elements subsets was generalized to r-elements subsets. The ranks and subdegrees of S_n acting on $X^{(r)}$ were computed where it was established that all suborbits of S_n on $X^{(r)}$ are self-paired. It was also shown that when $n \geq 2r$, the subdegrees are; 1, $r\binom{n-r}{r-1}, \binom{r}{2}\binom{n-r}{r-2}, \dots, \binom{n-1}{r}$ and the rank is $r+1$ (Nyaga *et al.*, 2011).

A method that uses a table of marks was devised to compute the subdegrees of transitive permutation groups (Ivanov *et al.*, 1983).

The subdegrees of all primitive permutation representations of $PSL(2, q)$ were computed by (Tchuda, 1986; Bon & Cohen, 1989) on the bases of the method proposed by (Ivanov *et al.*, 1983). The study was extended to the subdegrees of $PGL(2, q)$ on the cosets of maximal dihedral subgroups. It was shown that if $PSL(2, q)$ acts on the cosets of its maximal dihedral subgroup H, then the rank is at least $|G|/|H|^2$ and if $q > 100$, then the rank is greater than 5 (Faradzev and Ivanov, 1990). It was also established that when $PGL(2, q)$ acts on the cosets of its maximal dihedral subgroup of order $2(q-1)$ then its rank is $\frac{1}{2}(q+3)$ if q is odd and $\frac{1}{2}(q+2)$ if q is even. Consequently, the subdegrees are 1, $\frac{1}{2}(q-1)$, $2(q-1)$, and $(q-1)$ in $\frac{1}{2}(q-3)$ orbits if q is odd and 1, $2(q-1)$ and $(q-1)$ in $\frac{1}{2}(q-2)$ orbits when q is even (Kamuti, 2006).

The rank and subdegrees of the symmetric group S_n acting on ordered r-element subsets were calculated. It was shown that if $n \geq 6$, then the rank of S_n on $X^{[3]}$ is 34 and that of S_n on $X^{[2]}$ is 7 if $n \geq 4$. Particular cases when $r = 2$ and 3 were considered first and then a generalization for values of n such that $n \geq 2r$ was made. The subdegrees of S_n on $X^{[2]}$ were shown to be 1, 1, (n-2), (n-2), (n-2), (n-2), (n-2)(n-3). This was done using combinatorial arguments together with applicable theorems in the subject (Rimberia *et al.*, 2012a). The method has also been used in the study of the alternating group A_n acting on $X^{[r]}$ and $X^{(r)}$. According to Gachimu *et al*., (2015, 2016), the rank of A_n on $X^{[2]}$ is 7 for values of $n \geq 6$ and that of A_n on $X^{[3]}$ is 34 for values of $n \geq 8$. The study also generalized the rank of A_n on $X^{[r]}$ for all values of $n \geq 2(r+1)$. It was established that the rank of A_n on $X^{(r)}$ is $r+1$ when $n \geq 2r$. Additionally, the action of A_n on $X^{(r)}$ was proved to be transitive for all values of $n \geq r+1$ and imprimitive for values of $n=2r$.

On the other hand, Kimani *et al*. (2014) have used the method of marks in the computation of ranks and subdegrees of S_n acting on $X^{[2]}$ and $X^{[3]}$.

Subdegrees have been used to determine existence of distance-transitive graphs (Faradzev & Ivanov, 1990) and also classification of families of rank 3 permutation groups (Higman, 1970).

2.3 Suborbital graphs

A study on edge-colored graphs achieved a construction of the famous Petersen graph which has 10 vertices, 15 edges and girth 5 (Neumann, 1977).

The suborbital graph of the modular group Γ on $\widehat{\mathbb{Q}}$, the extended set of rationals ($\mathbb{Q} \cup \{\infty\}$) has been investigated. The action was defined through the fractional linear transformations of the upper half-plane by elements of Γ which constitutes the pairs of matrices $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(a, b, c, d, \in \mathbb{Z}, ad - bc = 1)$. It was shown that ${}^{r}/_{s}$ and ${}^{x}/_{y}$ are adjacent vertices if and only if, either $x \equiv ur \bmod n, y \equiv us \bmod n,$ and $ry - sx = n$ or $x \equiv -ur \bmod n,\ \ y \equiv -us \bmod n$ and $ry - sx = -n$ by Jones *et al*., (1991) where it was also conjectured that the suborbital graph is a forest if and only if it contains no triangles. The study was extended by Akbas (2001) where the conjecture was proved.

A method of constructing some of the suborbital graphs of $PSL(2, q)$ and $PGL(2, q)$ acting on the cosets of their maximal dihedral subgroups of orders (q-1) and $2(q-1)$ respectively was devised. This method gave an alternative way of constructing the Coxeter graph which has 28 vertices, 42 edges and girth 7 (Kamuti, 1992).

The circuits in the suborbital graphs of the normalizer of $\Gamma_0(N)$ on $\widehat{\mathbb{Q}}$, where N is a square-free positive integer were investigated. It was shown that any circuit in the suborbital graph $G(\infty, {}^{u}/_{n})$ of the normalizer of $\Gamma_0(N)$ is of the form;

$v \to T(v) \to T^2(v) \to \cdots T^{k-1}(v) \to v,$ where,$n > 1$, $v \in \widehat{\mathbb{Q}}$ and T an elliptic mapping of order k in the normalizer of $\Gamma_0(N)$ (Keskin, 2006).

The number of connected components of the graph, $G(0, x)$, of Γ_∞ on $\mathbb{Z}$ (the stabilizer of infinity in Γ acting on the set of integers) was found as $|x|$ (Kamuti *et al*., 2012).

2.4 Transitivity and primitivity

Investigations on the action of A_7 on $X^{(2)}$ established the existence of a primitive group of degree 21 which contains a non-abelian regular subgroup (Nagai, 1961).

Primitive rank 3 groups of even order in which the stabilizer has an orbit of prime length have been considered. It was shown that if *G* has no regular normal subgroup, then the minimal normal subgroup M of *G* is a simple group of rank 3 (Higman, 1966).

Cameron (1972) worked on multiply transitive permutation groups and also studied suborbits of primitive groups.

Transitivity of the actions of various subgroups of the modular group Γ on $\widehat{\mathbb{Q}}$ have been studied through the action defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \frac{x}{y} \to \frac{ax+by}{cx+dy}, \text{ for all } \frac{x}{y} \in \mathbb{Q} \text{ and } \infty \text{ represented as } \frac{1}{0} = \frac{-1}{0}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

A study on the suborbital graph of the modular group on $\widehat{\mathbb{Q}}$ established that the action was transitive and imprimitive. It was proved that the orbit containing infinity (∞) is $\widehat{\mathbb{Q}}$, and hence transitive (Jones *et al*., 1991).

Properties of suborbital graphs of the normalizer of $\Gamma_0(N)$ on $\widehat{\mathbb{Q}}$ were examined and it was established that the action was transitive and imprimitive. It was also proved that the stabilizer of a point was an infinite cyclic group (Akbas & Baskan, 1996).

A study on suborbital graphs of the congruence subgroup $\Gamma_0(N)$ on $\widehat{\mathbb{Q}}$, established that the action was not transitive and $\Gamma_0(p)$ was imprimitive on $\widehat{\mathbb{Q}}$ (Guler *et al.*, 2008).

Transitivity and primitivity of the dihedral group of degree 2^r ($r \geq 2$) were studied by Hamma and Aliyu (2010). It was shown that the group is transitive and imprimitive. However, not much has been studied on the group when the degree is not 2^r.

Investigations on the properties of suborbits and suborbital graphs of the symmetric group S_n acting on ordered r-element subsets revealed that S_n acts transitively on $X^{[3]}$ and that it acts imprimitively on $X^{[3]}$ provided $n > 4$ (Rimberia *et al*., 2012b).

The action of Γ_∞ on $\mathbb{Z}$ and the corresponding suborbital graphs was shown to be transitive and imprimitive (Kamuti *et al*., 2012).

Transitivity of A_n (n=5, 6, 7, 8) on $X^{(2)}$ and $X^{[2]}$ was established and properties of suborbital graphs for the action of A_n ($n \geq 5$) on $X^{(2)}$ were examined, where the non-trivial suborbital graphs were found to be connected and self-paired (Kinyanjui *et al*., 2013).

2.5 Summary

Several studies have been done on the actions of the subgroups of the modular group on the infinite set, $\widehat{\mathbb{Q}}$. The action of the finite group S_n on various sets has thoroughly been worked on. The action of the alternating group, a subgroup of S_n, on finite sets has also been considered. However, the dihedral and cyclic groups have not received much attention regarding the subject. This study generalizes the work on the dihedral group (Hamma and Aliyu, 2010), and also extends the investigations of the action of a cyclic subgroup of *PSL*(2, *q*) on subsets of $\mathbb{Z}$ (Kamuti *et al*., 2012)

CHAPTER THREE
MATERIALS AND METHODS

3.1 Introduction

This chapter gives a description of the materials and methods that were used to realize the intended results on the stated objectives. Section 3.2 captures the criterion which the study hinges on. Section 3.3 describes the method used to compute the rank and subdegrees of each action and section 3.4 addresses the requirements on construction of suborbital graphs.

3.2 Criterion

Let G be a group acting on a finite set X. The action is an equivalence relation and therefore partitions the set into equivalence classes, G-orbits. The techniques used require the G-orbit to be exactly 1 so that the action is transitive.

3.2.1 Transitivity

For each group G acting on X and $x \in X$, the size of the stabilizer $|G_x|$ was first established. Definitions 1.6.6 and 1.6.7 were applicable in this. Theorem 1.6.19 was then employed to compute the size of the orbit of x, $|orb_G(x)|$, in each of the actions. A comparison of $|orb_G(x)|$ and $|X|$ was used to determine transitivity of each action.

3.3 Rank and subdegrees

The fixed point set of every g $\in G_x$ on X was first identified. Next, Theorem 1.6.20 was employed to count the number of G_x-orbits on X, to provide the rank of G in each of the actions. The subdegrees were determined through the action of the elements of G_x on X. Using the definitions of paired suborbits, the property of pairedness was established. By Theorem 1.6.17, the number of self-paired suborbits was computed in each case.

3.4 Suborbital graphs

Using the stated definitions on suborbital graphs, alongside worked examples, the study formulated the respective theorems on construction of graphs associated to each of the actions.

CHAPTER FOUR
RESULTS AND DISCUSSION

This chapter discusses the results with reference to the stated objectives. Sections 4.1 to 4.5 has dealt with the action of the groups C_n and D_n on $X^{(r)}$, where C_n=<g>=<$(12\ldots n)$> and $X=\{1, 2, \ldots, n\}$. Section 4.6 and 4.7 discusses what was established on the action of C_n and D_n on $X^{[r]}$. The results have been given in form of theorems that were successfully proved.

4.1 Action of the groups C_n and D_n on $X^{(r)}$

The action of any group G on $X^{(r)}$ is defined by;
$h\{x_1, x_2, \ldots, x_r\}=\{h(x_1), h(x_2), \ldots, h(x_r)\}$, for all h in G and $\{x_1, x_2, \ldots, x_r\}$in $X^{(r)}$.

Theorem 4.1.1
The action of each of the groups, C_n and D_n, on $X^{(r)}$ is transitive if and only if $r=1$, $r=n-1$ or $r=n$.
Proof:

Let $G=C_n$ act on $X^{(r)}$ and suppose $h\in C_n$. Then h fixes an element in $X^{(r)}$, $r < n$, if and only if h is the identity. It follows, $|G_{\{1, 2, \ldots, r\}}|=1$. Using Theorem 1.6.19, $|orb_G\{1, 2, \ldots, r\}|=|G : G_{\{1, 2, \ldots, r\}}|=n$. If the action is transitive, then $|orb_G(\{1, 2, \ldots, r\})|=|X^{(r)}|$, $\Rightarrow n=\binom{n}{r}$, $\Rightarrow (n-1)!=(n-r)!r!$, $\Rightarrow r=1$ or $r=n-1$.
Secondly, let $G=D_n$, act on $X^{(r)}$ and suppose $h \in D_n$,. Then h fixes $\{1, 2, \ldots, r\}$ in $X^{(r)}$, $r < n$, if h is the identity or h is a reflection. It follows, $|G_{\{1, 2, \ldots, r\}}|=2$, $\Rightarrow|orb_G(\{1, 2, \ldots, r\})|=n$, by Theorem 1.6.19. If the action is transitive, then $|orb_G(\{1, 2, \ldots, r\})|=| X^{(r)}|$, $\Rightarrow n=\binom{n}{r}$, $\Rightarrow (n-1)!=(n-r)!r!$. Hence, $r=1$ or $r=n-1$. Conversely, if $r=1$ or $r=(n-1)$, then $|orb_G(\{1, 2, \ldots, r\})|=|X^{(r)}|$ and each of the actions is transitive.
Clearly, every $h \in G$ fixes 1 element in $X^{(n)}$. By Theorem 1.6.20, the action has 1 orbit and therefore transitive. However, the action on 1 element is trivial and the study concentrates on non-trivial actions. □

Theorem 4.1.2
The rank of C_n on X is n and it is equal to the rank of C_n on $X^{(n-1)}$. The length of each suborbit is 1 in each of the actions.

Proof:

Let $G=C_n$ and G_1 act on X. From Theorem 4.1.1, $h \in G_1$ is the identity and therefore fixes each element of X. The number of G_1-orbits on X is $n/1=n$, the rank of C_n on X. Clearly, the size of each suborbit is 1. The n suborbits of G on X are as follows; $\Delta_0=\{1\}$, $\Delta_1=\{2\}$, …, $\Delta_{n-1}=\{n\}$, where $\Delta_i=\{i+1\}$, $i=0, 1, \ldots, n-1$.

Secondly, $h \in G_1, \Longrightarrow h \in G_{\{2, \ldots, n\}}$ and therefore the $G_{\{1, 2, \ldots, n-1\}}$ –orbits are as follows; $\Delta_0=\{1, 2, \ldots, n-1\}$, $\Delta_1=\{2, 3, \ldots, n-1, n\}$, …, $\Delta_{n-1}=\{n, 1, 2, \ldots, n-2\}$, where $\Delta_i=\{i+1, i+2, \ldots, i-1\}$. Clearly, the subdegrees are; 1, 1, …, 1 (n ones).

Theorem 4.1.3

Let C_n act on X. Then suborbits Δ_i and Δ_j are paired if and only if $i+j=0$ *mod n*.

Proof:

Suppose Δ_i and Δ_j are paired suborbits of $C_n=< g >=< 12\ldots n >$. Then there exists g^k in C_n, such that $g^k\Delta_0=\Delta_j$ and $g^k\Delta_i=\Delta_0$, from Definition 1.6.9. It follows, $1+k=j+1$ and $i+1+k=1$ *mod n*, $\Longrightarrow i+j=0$ *mod n*. Conversely, if $i+j=0$ *mod n*, then, $g^j\Delta_0=\Delta_j$ and $g^j\Delta_i=\Delta_0$. Hence, Δ_i and Δ_j are paired suborbits.

Corollary 4.1.4

Let C_n act on X. Then Δ_i is self- paired if and only if i=0 or i=n/2 mod n.

Proof:

From Theorem 4.1.3, Δ_i is self- paired if and only if $i=j$ in the equation, $i+j=0$ *mod n*. It follows, $i=0$ or $i=n/2$ *mod n*.

Theorem 4.1.5

The suborbits Δ_i and Δ_j are paired in the action of C_n on $X^{(n-1)}$ if and only if $i+j=0$ *mod n*.

Proof:

Suppose Δ_i and Δ_j are paired suborbits of C_n. Then there exists g^k in C_n such that $g^k\Delta_0=\Delta_j$ and $g^k\Delta_i=\Delta_0$, by definition of pairedness. It follows, $1+k=j+1$ and $i+1+k=1$, $2+k=j+2$ and $i+2+k=2$, …, $n-1+k=j-1$ and $i-1+k=n-1$ *mod n*. Hence, $i+j=0$ *mod n*. Conversely, if $i+j=0$ *mod n*, then $g^j\Delta_0=\Delta_j$ and $g^j\Delta_i=\Delta_0$. It follows, Δ_i and Δ_j are paired.

Corollary 4.1.6

The suborbit Δ_i is self- paired in the ation of C_n on $X^{(n-1)}$ if and only if i=0 or i=n/2.

Proof:

Let Δ_i be a self- paired suborbit of C_n. Then, $i=j \bmod n$, in the equation, $i+j=0$, from Theorem 4.1.5. This is possible if and only if i=0 or $i=n/2 \bmod n$.

Theorem 4.1.7

The number of self- paired suborbits of C_n on X is 1 when n is odd and 2 when n is even.

Proof:

Let $x \in X$ and $h \in C_n$. When n is odd, h^2 fixes x if h is the identity. Thus, $\sum_{h\in C_n} |Fix(h^2)| = n$. By Theorem 1.6.17, , the number of self -paired suborbits is $\frac{1}{n}(n) = 1$. This corresponds to the trivial suborbit.

When n is even, h^2 fixes $x \in X$ if h is the identity or h is a rotation of 180^0. It follows, $\sum_{h\in C_n} |Fix(h^2)| = 2n$, and the number of self- paired suborbits is $\frac{1}{n}(2n) = 2$. The self-paired suborbits of C_n, in this case, are Δ_0 and $\Delta_{n/2}$.

Theorem 4.1.8

The number of self- paired suborbits of C_n acting on $X^{(n-1)}$ is 1 when n is odd and 2 when n is even.

Proof:

Let $A \in X^{(n-1)}$ and $h \in C_n$. Now, h^2 fixes A if and only if h^2 is the identity. When n is odd, this is possible only if h is the identity. Thus, $\sum_{h\in C_n}(|Fix(h^2\ )|) = n$. By Theorem 1.6.17, the number of self -paired suborbits is $\frac{1}{n}(n) = 1$. This corresponds to the trivial suborbit.

When n is even, h^2 fixes A if h is the identity or h is a rotation of 180^0. It follows, the number of self- paired suborbits is $\frac{1}{n}(2n) = 2$. The 2 suborbits are Δ_0 and $\Delta_{n/2}$.

Theorem 4.1.9

The rank of D_n on X equals the rank of D_n on $X^{(n-1)}$. The rank is $\frac{n+1}{2}$ when n is odd and $\frac{n+2}{2}$ when n is even.

Proof:

Let G=D_n and G_1, the stabilizer of 1 in G, act on X. From Theorem 4.1.1, $h \in G_1$ is either the identity or a reflection. When n is odd, the identity h fixes n elements in X and the reflection h fixes 1 element. Thus, $\sum_{h\in G_1}|Fix(h)|$=n+1. Using Theorem 1.6.20, the number of G_1-orbits on X is $\frac{n+1}{2}$, the rank of D_n on X. Next, the identity fixes n elements in $X^{(n-1)}$. From the reflection $h \in G_1$, $h(1)=1 \Rightarrow h\{2, 3, \ldots, n\}=\{2, 3, \ldots, n\} \in X^{(n-1)}$. Hence, the number of $G_{\{2, 3, \ldots, n\}}$-orbits on $X^{(n-1)}$ is $\frac{n+1}{2}$, the rank of D_n on $X^{(n-1)}$.

When n is even, the identity fixes n elements in X and the reflection fixes 2 elements. It follows, $\sum_{h\in G_1}|Fix\ (h)|$=$n$+2 and the number of G_1-orbits on X is $\frac{n+2}{2}$. Now, $h \in G_1$ is such that $h(1)=1$ and $h((n+2)/2)=(n+2)/2$, $\Rightarrow$ $h\{2, 3, \ldots, n\}=\{2, 3, \ldots, n\} \in X^{(n-1)}$ and $h(X|\{(n+2)/2\})=X|\{(n+2)/2\}$. The identity h fixes $\binom{n}{n-1}$=n elements in $X^{(n-1)}$. It follows, $\sum_{h\in G_{\{2,\ldots n\}}}|Fix\ (h)|$=$n$+2 and the number of $G_{\{2, 3, \ldots, n\}}$-orbits on $X^{(n-1)}$ is $\frac{n+2}{2}$, the rank of D_n on $X^{(n-1)}$. □

Theorem 4.1.10

The suborbits of D_n on X are of the form;

$\Delta_i=\{i+1, n+1-i\}$, where i=0, 1, …, $\frac{n-1}{2}$, when n is odd, i=0, 1, …, $\frac{n}{2}$, when n is even.

Proof:

Let G_1 be the stabilizer of 1 in D_n. When n is odd, $G_1=\{1, (2\ n)(3\ n\text{-}1)\ldots(\frac{n+1}{2}\ \frac{n+3}{2})\}$. Now, $h \in G_1$ is such that $hx = x$ or $hx = n+2-x$ for all $x \in X$. If Δ_i is the G_1-orbit of i+1, then the suborbits of D_n

are as follows;

$\Delta_0=\{1\}$, $\Delta_1=\{2, n\}$, $\Delta_2=\{3, n-1\}$, …, $\Delta_{\frac{n-1}{2}}=\left\{\frac{n+1}{2}, \frac{n+3}{2}\right\}$, where $\Delta_i=\{i+1, n+1-i\}$, i=0, 1, …, $\frac{n-1}{2}$.

When n is even, G_1={1, (2 n)(3 n-1)…$\left(\frac{n}{2}\ \frac{n+4}{2}\right)$}. The suborbits of D_n are then as follows;

Δ_0={1}, Δ_1={2, n}, …, $\Delta_{\frac{n}{2}}$= {$\frac{n+2}{2}$}, where Δ_i ={i+1, n+1-i}, i=0, 1, …, $\frac{n}{2}$.

Theorem 4.1.11

The suborbits of D_n on $X^{(n-1)}$ are of the form;

Δ_i = {{i+1, i+2, …, i-1}, {1-i, 2-i, …, n-1-i}}, (i = 0, 1, …, $\frac{n-1}{2}$) when n is odd, and

Δ_i={{i+1, i+2, …, i-1}, {1-i, 2-i, …, n-1-i}},(i=0, 1, …, $\frac{n}{2}$) when n is even.

Proof:

Let D_n act on X. When n is odd, G_n={1, (1 n-1)(2 n-2)…$\left(\frac{n-1}{2}\ \frac{n+1}{2}\right)$}=$G_{\{1, 2, \ldots, n-1\}}$. The $G_{\{1, 2, \ldots, n-1\}}$–orbit of $A \in X^{(n-1)}$ is given by {A, {n-x}| $x \in A$}. If Δ_i is the orbit of {i+1, i+2, …, i-1}, then Δ_i={{i+1, i+2, …, i-1}, {1-i, 2-i, …, n-1-i}} (i=0,1, …, $\frac{n-1}{2}$).

Table 1: Subdegrees of D_n acting on $X^{(n-1)}$ when n is odd

.

Suborbit length	1	2
Number of suborbits	1	$\frac{n-1}{2}$

When n is even, the suborbits of D_n are then as follows;

Δ_0={1, 2, …, n-1}, Δ_1={{2, 3, …, n}, {n, 1, …, n-2}}, …, $\Delta_{\frac{n}{2}}$={$\frac{n}{2}$+1, $\frac{n}{2}$+2, …, $\frac{n}{2}$ -1}. The suborbits have the general form; Δ_i={{i+1, i+2, …, i-1}, {1-i, 2-i, …, n-1-i}}, i=0, 1, …, $\frac{n}{2}$.

Table 2: Subdegrees of D_n acting on $X^{(n-1)}$ when n is even

Suborbit length	1	2
Number of suborbits	2	$\frac{n-2}{2}$

Theorem 4.1.12

The action of D_n on X has exactly 1 suborbit of length 1 and (n-1)/2 suborbits of length 2 when n is odd. But there are 2 suborbits of length 1 and (n-2)/2 suborbits of length 2 when n is even.

Proof:

Let G_1 act on X and $x \in X$. From Theorem 4.1.10, the G_1-orbit of x is $\{x, n+2-x\}$. The elements in a suborbit of length 1 are given by solving $2x=2 \ mod\ n$. It follows, $x=1$ when n is odd. However, $x=1$ or $(n+2)/2 \ mod\ n$ when n is even. The numbers of the remaining follow from Theorem 4.1.9.

Example 4.1.13

Let D_{10} act on X. The six G_1-orbits on X are;

$\Delta_0=\{1\}$, $\Delta_1=\{2, 10\}$, $\Delta_2=\{3, 9\}$, $\Delta_3=\{4, 8\}$, $\Delta_4=\{5, 7\}$, $\Delta_5=\{6\}$.

Theorem 4.1.14

The number of self-paired suborbits of D_n on X is $\frac{n+1}{2}$ when n is odd and $\frac{n+2}{2}$ when n is even.

Proof:

Let D_n act on X and $x \in X$. When n is odd, $h^2 \in D_n$ fixes x if h is the identity or a reflection. From the identity, h^2 fixes n elements in X. From each reflection, h^2 fixes n elements. The number of elements fixed by n reflections is n^2. By Theorem 1.6.16, the number of self-paired suborbits is $\frac{1}{2n}(n^2+n)=\frac{n+1}{2}$.

When n is even, h^2 fixes x in X if h is the identity, or a reflection or a rotation of 180^0. The identity h^2 fixes n elements in X, each reflection h^2 fixes n elements in X. The n reflections fix n^2 elements and the rotation h^2 fixes n elements. The number of self-paired suborbits is $\frac{1}{2n}(n^2+2n)=\frac{n+2}{2}$. □

Theorem 4.1.15

The number of self-paired suborbits of D_n on $X^{(n-1)}$ is $\frac{n+1}{2}$ when n is odd, and $\frac{n+2}{2}$ when n is even.

Proof:

From Theorem 4.1.9, $G_1=G_{\{2, 3, \ldots, n\}}$. The action of $h \in G_1$ on X, induces the action of $G_{\{2, 3, \ldots, n\}}$ on $X^{(n-1)}$. The number of self-paired suborbits of D_n on $X^{(n-1)}$ follows from Theorem 4.1.14.

Corollary 4.1.16

All suborbits of D_n on X are self-paired.

Proof:

From Theorems 4.1.9 and 4.1.14, the number of suborbits equals the number of self-paired suborbits. Hence, the proof.

Corollary 4.1.17

All suborbits of D_n on $X^{(n-1)}$ are self-paired.

Proof:

The proof follows from Theorem 4.1.9 and Theorem 4.1.15.

Remark 4.1.18

The suborbits Δ_i and Δ_{n-i} of D_n on X are the same.

Proof:

From Theorem 4.1.10, $\Delta_{n-i}=\{n-i+1, n+1-(n-i)\}=\{n+1-i, i+1\}=\Delta_i$.

4.2 Suborbitals and suborbital graphs of C_n acting on X

Theorem 4.2.1

Let O_i be the suborbital of G corresponding to Δ_i. Then $(c, d) \in O_i(1, i+1)$ if and only if $d\text{-}c=I$ *mod* n.

Proof:

Suppose $(c, d) \in O_i$. Then $(c, d)=g^j(1, i+1)$, $\Longrightarrow c=g^j(1)$ and $d=g^j(i+1)$, $\Longrightarrow c=1+j$ *mod* n and $d=i+1+j$ *mod* n. Therefore, $d\text{-}c=i$ *mod* n. Conversely, suppose (c, d) is such that $d\text{-}c=i$ *mod* n. Then, $g^{(c-1)}(1, i+1)=(c, c+i)=(c, d)$. Therefore, $(c, d) \in O_i$.

4.2.1 Suborbital graphs of C_6 acting on X

The graph Γ_1 has an edge from c to d if and only if $d\text{-}c=1$ *mod* 6. The graph is shown in Figure 1 below.

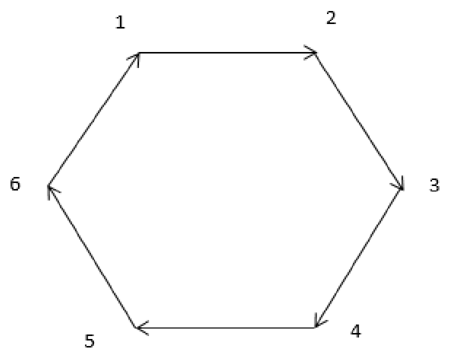

Figure 1: The graph Γ_1 of C_6 on X

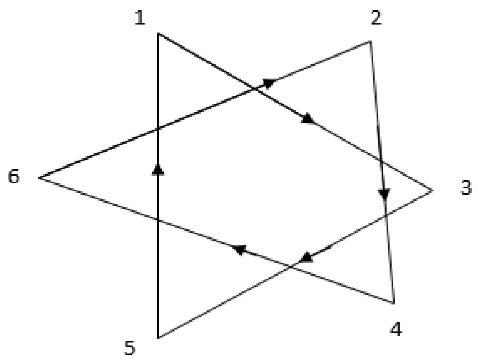

Figure 2: The graph Γ_2 of C_6 on X

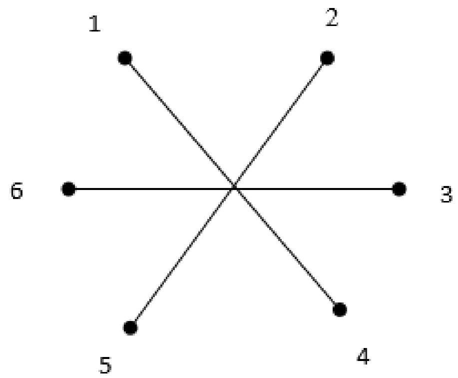

Figure 3: The graph Γ_3 of C_6 on X

The graphs Γ_4 and Γ_5 are paired with Γ_2 and Γ_1 respectively.

4.2.2 Suborbital graphs of C_7 acting on X

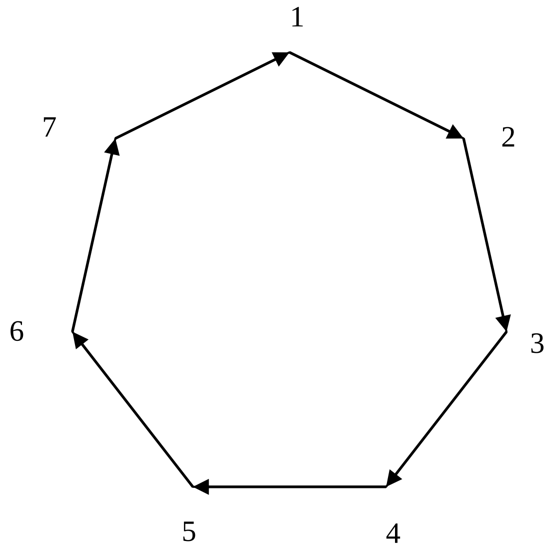

Figure 4: The graph Γ_1 of C_7 on X

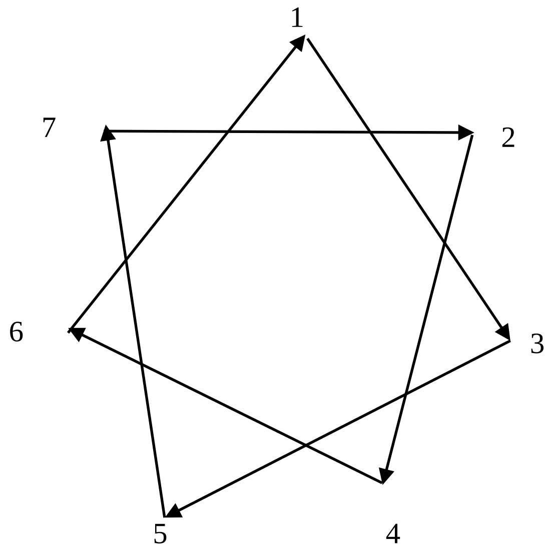

Figure 5: The graph Γ_2 of C_7 on X

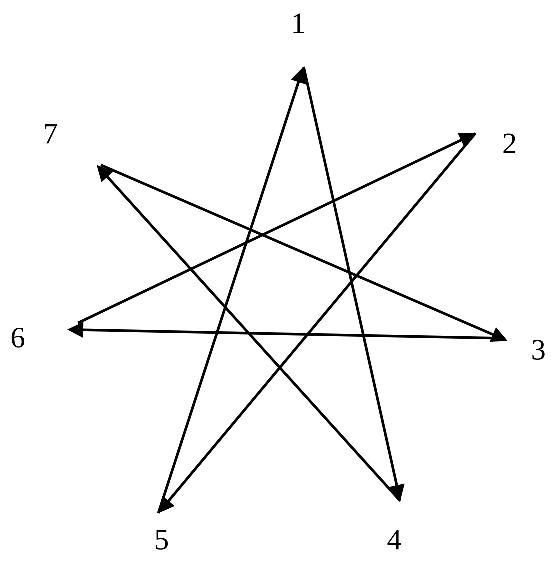

Figure 6: The graph Γ_3 of C_7 on X

The graphs Γ_4, Γ_5 and Γ_6 are paired with Γ_3, Γ_2 and Γ_1 respectively.

4.2.3 Suborbital graphs of C_8 acting on X

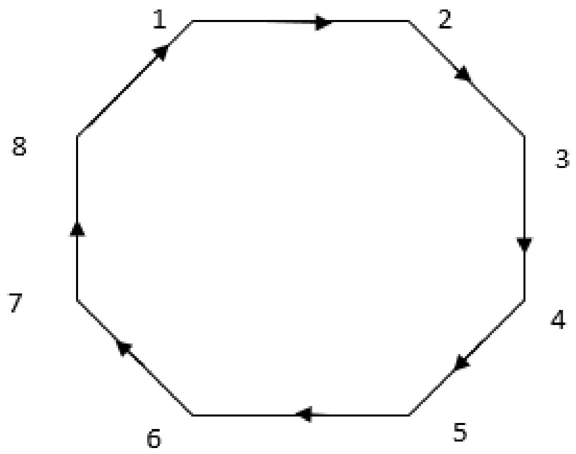

Figure 7: The graph Γ_1 of C_8 on X

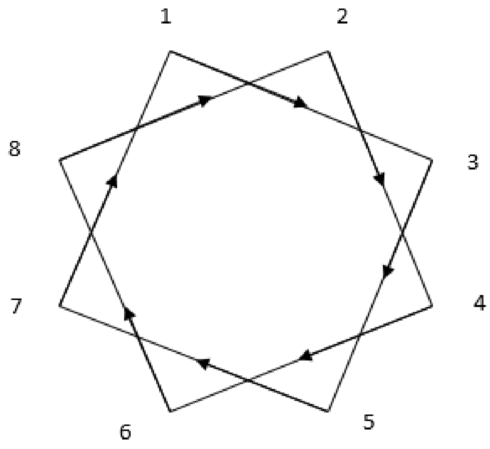

Figure 8: The graph Γ_2 of C_8 on X

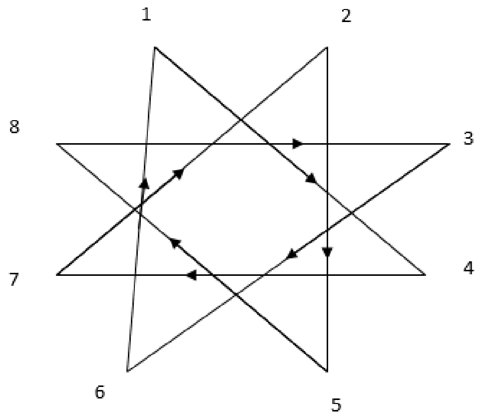

Figure 9: The graph Γ_3 of C_8 on X

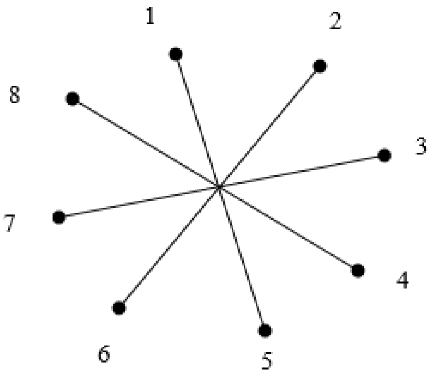

Figure 10 : The graph Γ_4 of C_8 on X

The graphs Γ_5, Γ_6, and Γ_7 are paired with Γ_3, Γ_2, and Γ_1 respectively.

4.2.4 Suborbital graphs of C_{10} on X

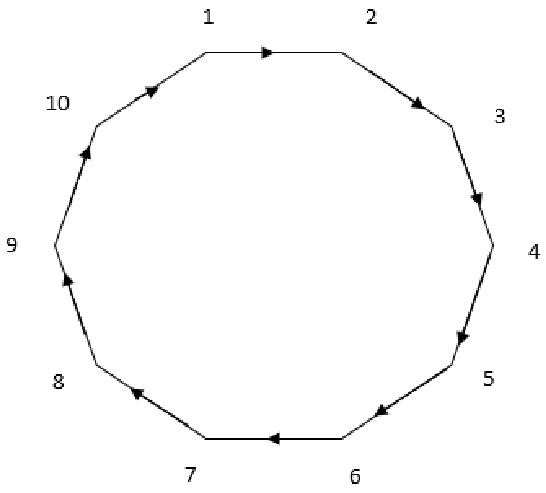

Figure 11: The graph Γ_1 of C_{10} on X

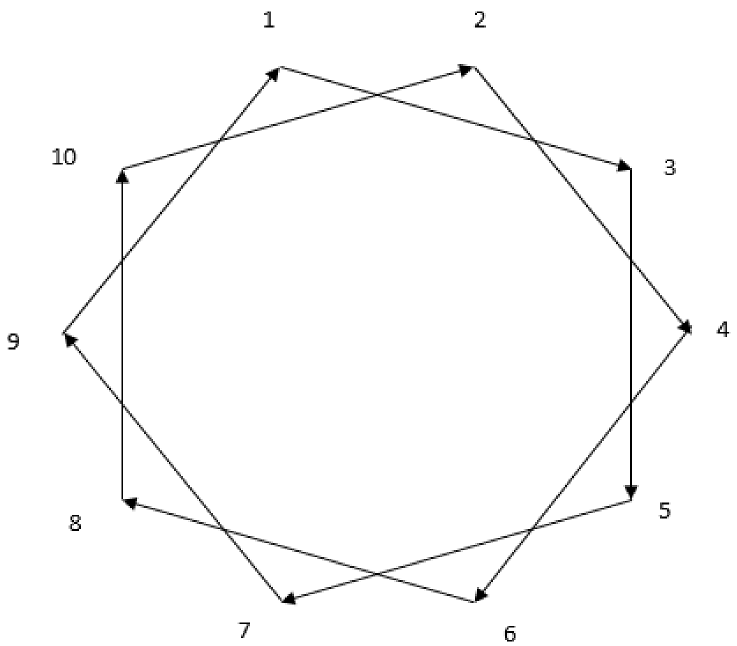

Figure 12: The graph Γ_2 of C_{10} on X

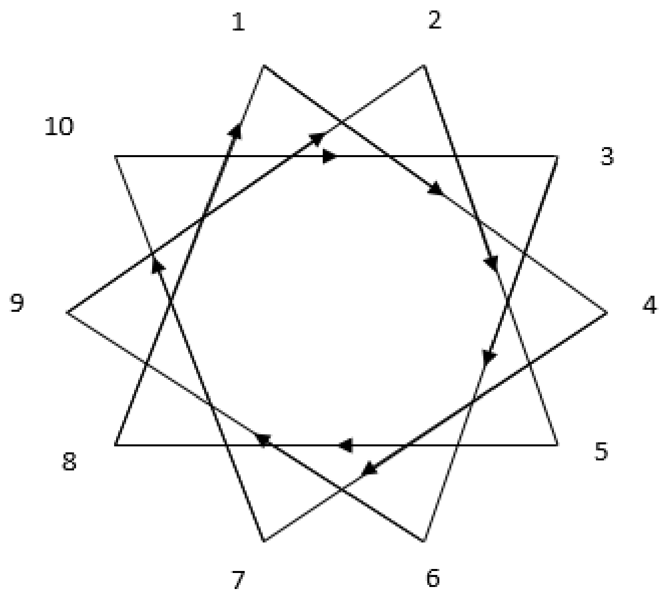

Figure 13: The graph Γ_3 of C_{10} on X

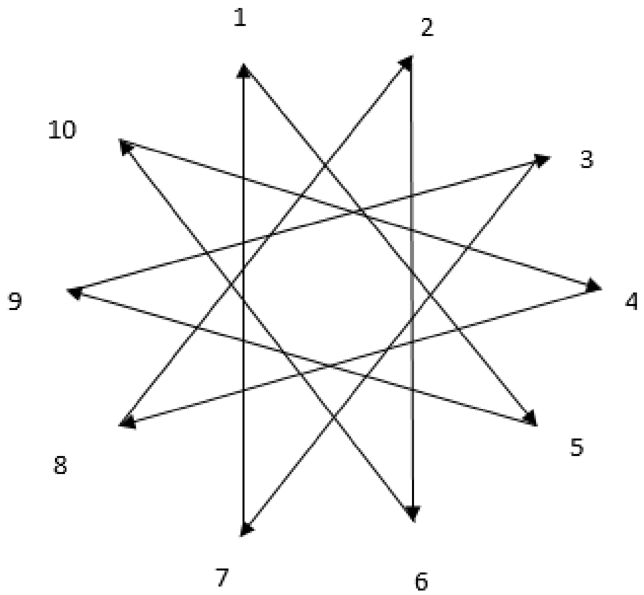

Figure 14: The graph Γ_4 of C_{10} on X

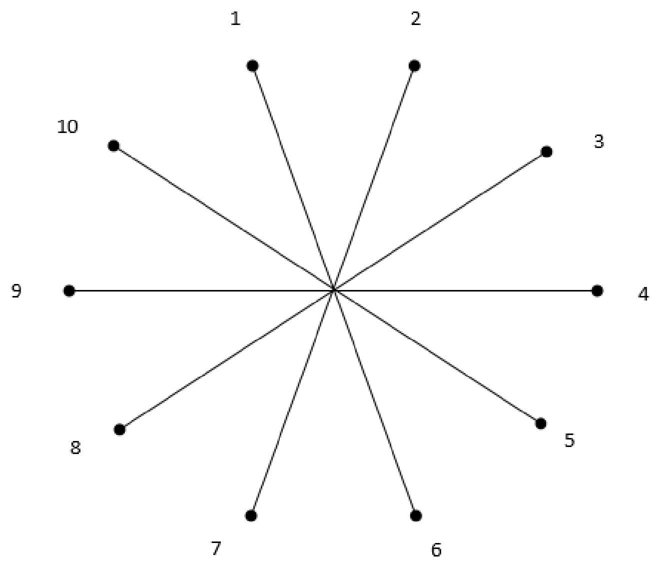

Figure 15: The graph Γ_5 of C_{10} on X

The graphs Γ_6, Γ_7, Γ_8 and Γ_9 are paired with Γ_4, Γ_3, Γ_2 and Γ_1 respectively.

4.2.5 Properties of suborbital graphs of C_n on X

Theorem 4.2.2

The graph Γ_i is connected if and only if i and n are coprime.

Proof:

The graph Γ_i has an edge from c to d if and only if $d- c=i \bmod n$. The cycles of Γ_i correspond to the cycles of g^i, where $g=(12\ldots n)$. The graph is connected only if g^i consists of 1, n-cycle. From the theory of cyclic groups, this is possible if and only if i and n are coprime.

Theorem 4.2.3

The girth of the suborbital graph Γ_i is n/d, where d is gcd(i, n), provided that $i \neq n/2$.

Proof:

The girth of the graph Γ_i is the smallest value of k such that $g^k=1$. If i and n have s common divisors, then the smallest value of $k = n/d$, where d is gcd(i, n). If $i=n/2$, then g^i consists of $n/2$, 2-cycles. Thus, an edge joining exactly 2 vertices is a connected component and the graph has no cycles.

Theorem 4.2.4

The number of connected components in Γ_i is d, the gcd of i and n.

Proof:

From Theorem 4.2.2, the cycles of Γ_i correspond to the cycles of g^i. From the theory of cyclic groups, g^i consists of d, n/d-cycles. It follows that d is the number of connected components in Γ_i

Theorem 4.2.5

The suborbital graph Γ_i is undirected if and only if $i=n/2$.

Proof:

Suppose Γ_i is undirected. Then, by Theorem 4.2.1, there is an edge from c to d and one from d to c. It follows, $d\text{-}c=i$ and $c\text{-}d=i \bmod n$, $\Longrightarrow i=n/2 \bmod n$. Conversely, if $i=n/2 \bmod n$, then $c\text{-}d=n/2$ and $d\text{-}c=n\text{-}n/2=n/2$. Therefore, Γ_i is undirected.

Theorem 4.2.6

The graphs Γ_i and Γ_j are paired if and only if $i+j=0 \bmod n$.

Proof:

Suppose Γ_i and Γ_j are paired, and there is an edge from c to d in Γ_i. Then $c\text{-}d=i$ and $d\text{-}c=j \bmod n$, from the definition of paired graphs. It follows, $i+j=0 \bmod n$. Conversely, if $i+j=0 \bmod n$, then the edge c to d of Γ_i is in the opposite direction of the edge c to d of Γ_{-i}.

Theorem 4.2.7

All non-trivial suborbital graphs corresponding to the action of C_n on X are directed if and only if n is odd.

Proof:

From Theorem 4.2.6, Γ_i is undirected if and only if $i=n/2$. This is possible only if n is even. It follows that Γ_i is directed for all odd n.

Theorem 4.2.8

The action of C_n on X is primitive if n is prime.

Proof:

Let Γ_i be the graph of C_n on X, where n is prime. Then Γ_i is connected if and only if i and n are coprime, by Theorem 4.2.2. It follows, every non-trivial Γ_i is connected and the action is primitive by Theorem 1.6.16.

Theorem 4.2.9

The number of connected graphs Γ_i corresponding to the action of C_n on X is $\phi(n)$, the totient phi function.

Proof:

From Theorem 4.2.2, the number of connected graphs Γ_i is the number of positive integers less than or equal to n that are coprime to n. The number is $\phi(n)$.

4.3 Suborbitals and suborbital graphs of C_n acting on $X^{(n-1)}$

The suborbital O_i corresponding to suborbit Δ_i is the set $O_i=\{h(\Delta_0, \Delta_i) \mid h \in G\}$, where $\Delta_i=\{i+1, i+2, \ldots, i-1\}$. The suborbital graph Γ_i corresponding to Δ_i is constructed by considering $X^{(n-1)}$ as the vertices, and drawing an edge from C to D if and only if $(C, D) \in O_i$.

Theorem 4.3.1

Let O_i be the suborbital of C_n corresponding to Δ_i. Suppose (C, D) is such that c_k and d_k are the kth coordinates of C and D respectively. Then $(C, D) = (\{c_1, c_2, \ldots, c_{n-1}\}, \{d_1, d_2, \ldots, d_{n-1}\})$ is in O_i (Δ_0, Δ_i) if and only if $d_k - c_k = i \bmod n$, for all $k=1, 2, \ldots, n-1$.

Proof:

If (C, D) is in O_i, then $(C, D)=g^j(\Delta_0, \Delta_i)$, $\Longrightarrow c_k=k+j$ and $d_k=i+k+j$. Therefore, $d_k-c_k=i \bmod n$. Conversely, suppose (C, D) is such that $d_k-c_k= i \bmod n$. Then $g^i(\Delta_0, \Delta_i)=(C, D)$, $\Longrightarrow (C, D) \in O_i$.

4.3.1 Suborbital graphs of C_5 on $X^{(4)}$

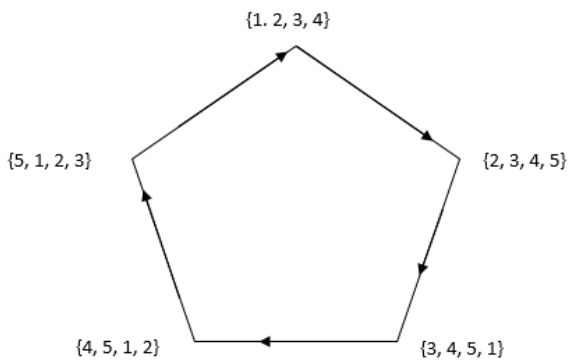

Figure 16: The graph Γ_1 of C_5 on $X^{(4)}$

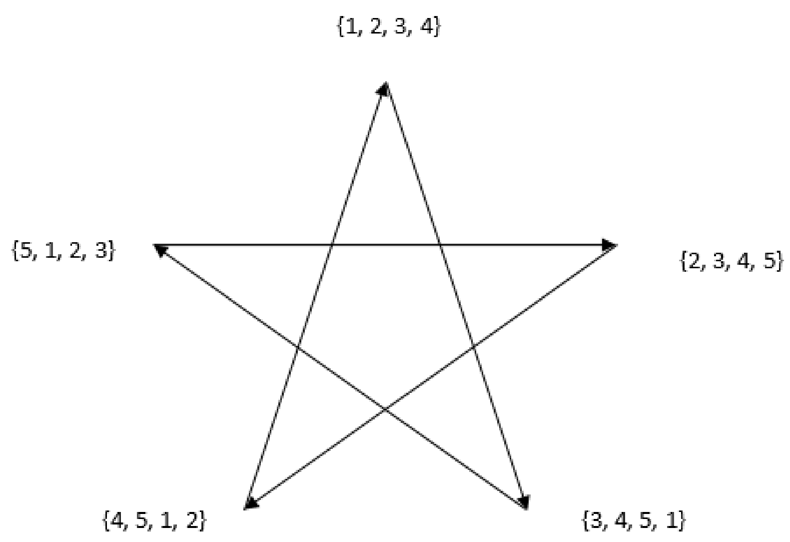

Figure *17:* The graph Γ_2 of C_5 on $X^{(4)}$

The graphs Γ_3 and Γ_4 are paired with Γ_2 and Γ_1 respectively.

4.3.2 Suborbital graphs of C_8 on $X^{(7)}$

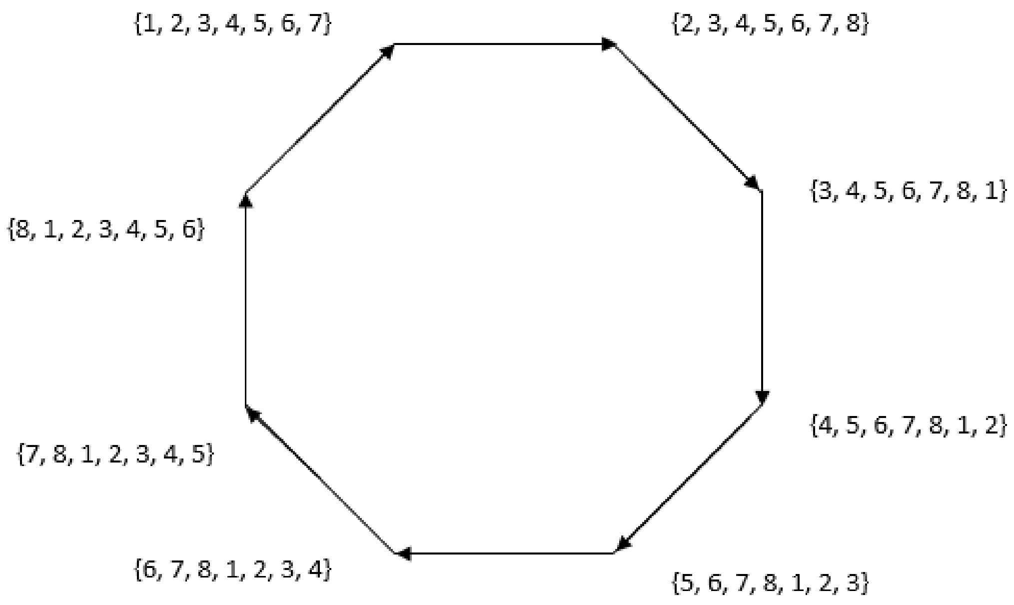

Figure 18: The graph Γ_1 of C_8 on $X^{(7)}$

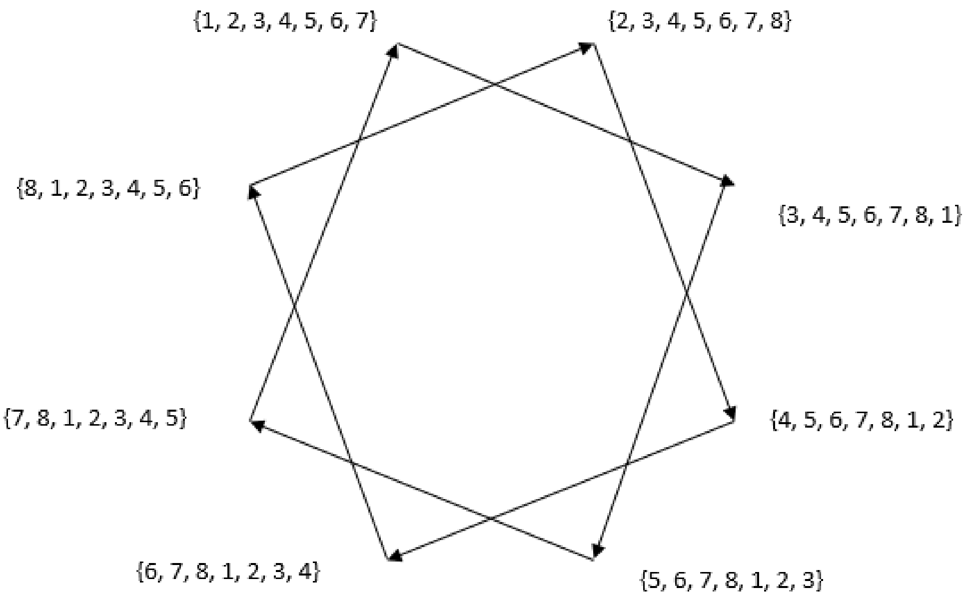

Figure 19: The graph Γ_2 of C_8 on $X^{(7)}$

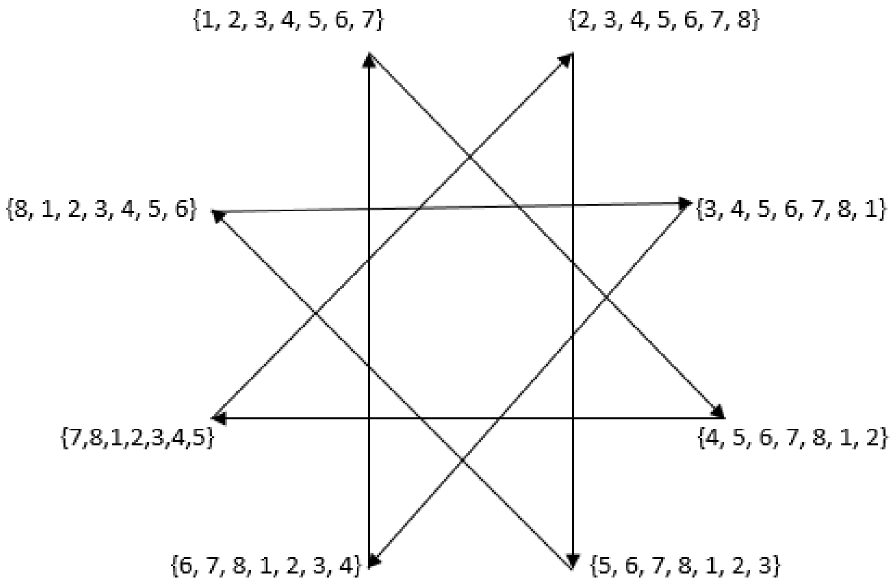

Figure 20: The graph Γ_3 of C_8 on $X^{(7)}$

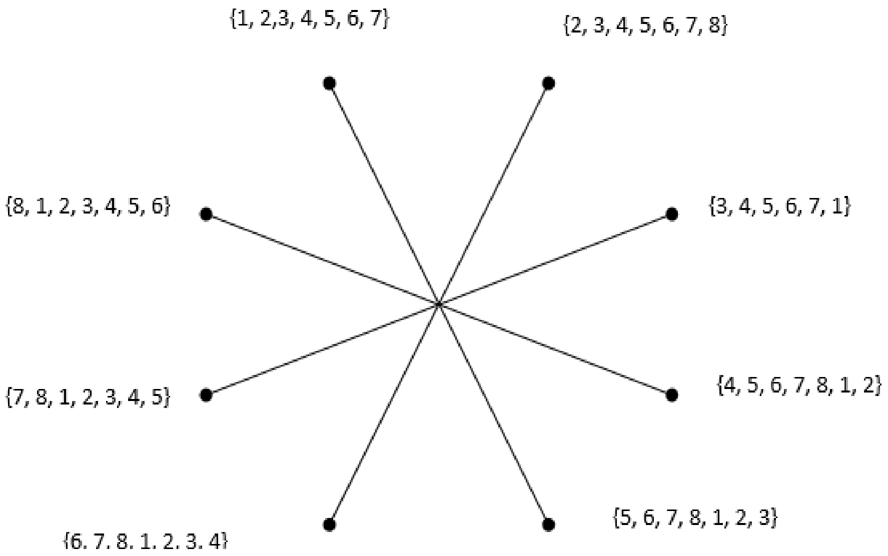

Figure 21: The graph Γ_4 of C_8 on $X^{(7)}$

Theorem 4.3.2

If i and n are coprime, then the suborbital graph Γ_i corresponding to O_i (Δ_0, Δ_i) is isomorphic to the graph shown in Figure 22 below.

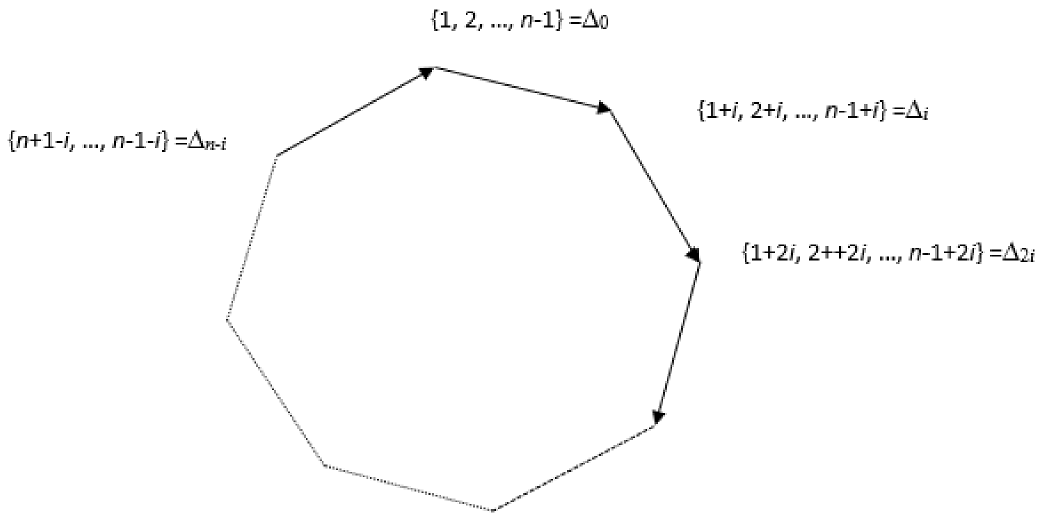

Figure 22: The graph Γ_i of C_n on $X^{(n-1)}$ when *i* and *n* are coprime

Proof:

From Theorem 4.3.1, Γ_i has an edge from C to D if and only if the k^{th} coordinate at D is $i \bmod n$ more than the k^{th} coordinate at C. The graph is connected if i and n are coprime, from Theorem 4.2.2. It follows, the graph is as shown in Figure 22.

Theorem 4.3.3

If i and n have s common divisors, and gcd(i, n) is d, then Γ_i is isomorphic to the graph in Figure 23 below.

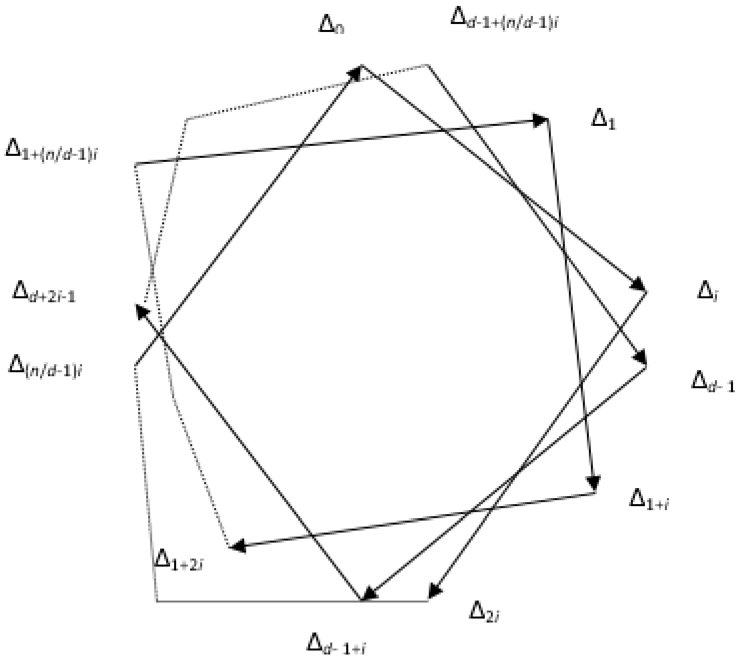

Figure 23: The graph Γ_i of C_n on $X^{(n-1)}$ when gcd(i, n)=d, $d > 1$

Proof:

From Theorem 4.3.1, Theorem 4.2.3, and Theorem 4.2.4, the graph has d connected components and girth n/d. The cycles of Γ_i are of the form;

$(1\ 1+i\ \ldots\ 1+(n/d-1)i)(2\ 2+i\ \ldots\ 2+(n/d-1)i)\ldots(d\ d+i\ \ldots\ d+(n/d-1)i)$, and the graph appears in Figure 23.

Theorem 4.3.4

If $i=n/2$, then the graph Γ_i is shown in Figure 24 below.

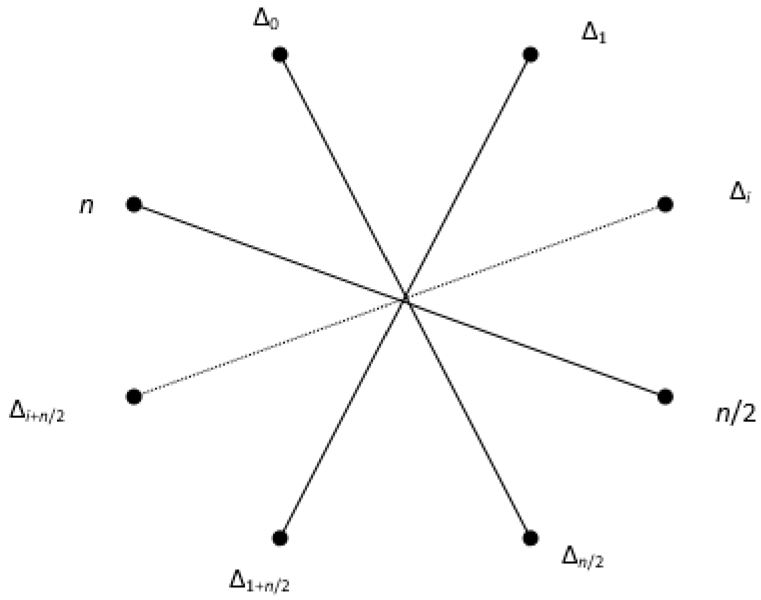

Figure 24: The graph Γ_i of C_n on $X^{(n-1)}$ when $i=n/2$

Proof:

From Theorem 4.3.1, Γ_i has cycles of the form; $(1\ 1+n/2)(2\ 2+n/2)\ldots(n/2\ n)$. It appears as shown in Figure 24.

Theorem 4.3.5

The action of C_n on X is equivalent to the action of C_n on $X^{(n-1)}$.

Proof:

Using Theorem 1.6.21, let (G_1, X) and $(G_2, X^{(n-1)})$ be the action of C_n on X and the action of C_n on $X^{(n-1)}$, respectively. Let ϕ: $G_1 \to G_2$ such that $\phi(h)=h$, for all $h \in G_1$. Define Θ: $X \to X^{(n-1)}$ such that $\Theta(x)=X|x$, for all $x \in X$. Now, $\Theta(hx)= X|hx=h(X|x)= \phi(h)\ \Theta(x)$. □

The equivalence exhibited by the two actions offers facilities for determining the properties of Γ_i, in the action of C_n on $X^{(n-1)}$, through the properties of Γ_i in the action of C_n on X. Hence, the properties of Γ_i in the action of C_n on $X^{(n-1)}$ follow from Section 4.2.5.

4.4 Suborbitals and suborbital graphs of D_n acting on X

Theorem 4.4.1

Suppose O_i is the suborbital corresponding to Δ_i. Then $(c, d) \in O_i(1, i+1)$ if and only if $d\text{-}c=i \bmod n$ or $d\text{-}c=n\text{-}i \bmod n$.

Proof:

If $(c, d) \in O_i$, then $(c, d)=hg^j(1, i+1)$, where $h(c, d)=(d, c)$ and $g=(12\ldots n)$. Now, $(c, d)=(1+j, i+1+j)$ or $(i+1+j, 1+j) \Longrightarrow d\text{-}c=i \bmod n$ or $n\text{-}i \bmod n$. Conversely, if $d\text{-}c=i \bmod n$ or $d\text{-}c= n\text{-}i \bmod n$, then $g^{(c-1)}(1, i+1)=(c, d)$. Since Δ_i is self-paired, there exists h in D_n such that $h(c, d)=(d, c)$, $\Longrightarrow (c, d) \in O_i$.

4.4.1 Suborbital graphs of D_6 on X

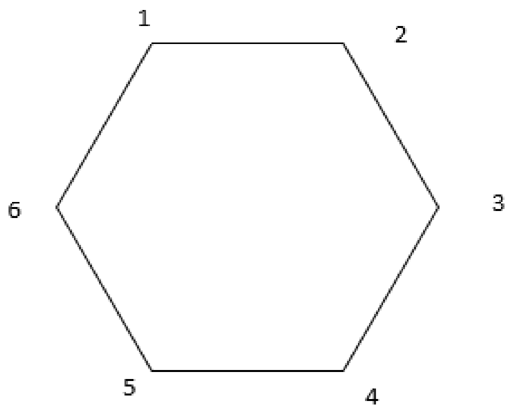

Figure 25: The graph Γ_1 of D_6 on X

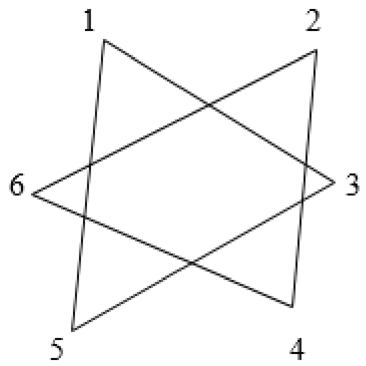

Figure 26: The graph Γ_2 of D_6 on X

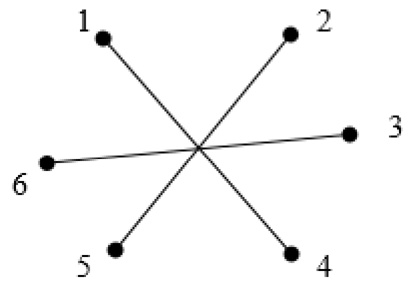

Figure 27: The graph Γ_3 of D_6 on X

4.4.2 Suborbital graphs of D_7 acting on X

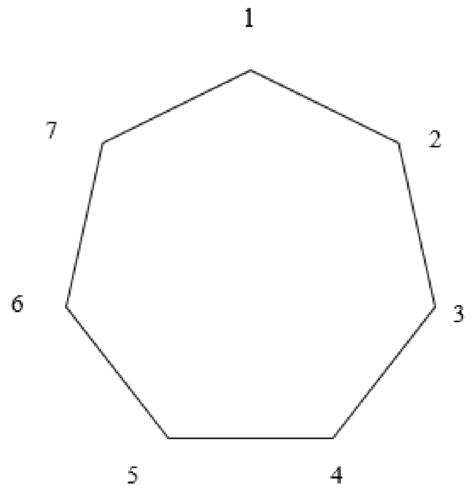

Figure 28: The graph Γ_1 of D_7 on X

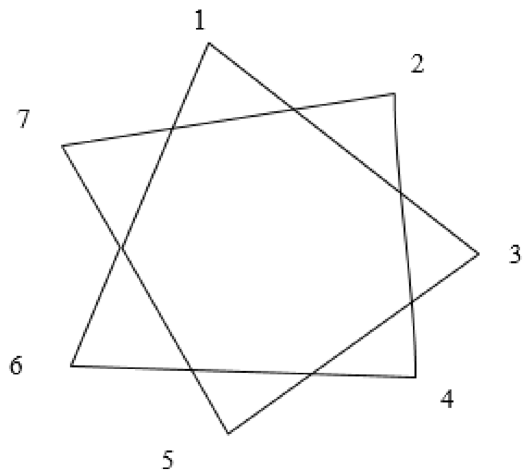

Figure 29: The graph Γ_2 of D_7 on X

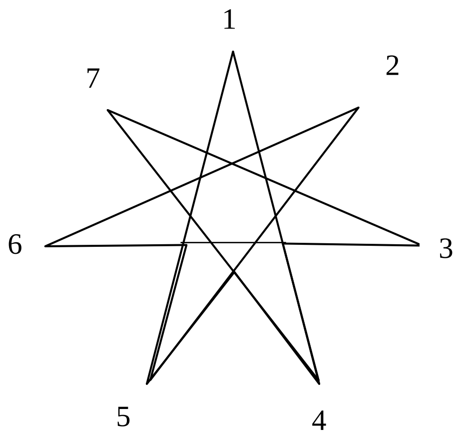

Figure 30: The graph Γ_3 of D_7 on X

4.4.3 Suborbital graphs of D_8 acting on X

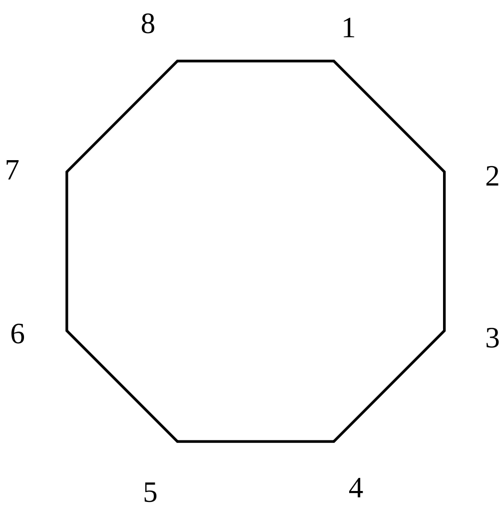

Figure 31: The graph Γ_1 of D_8 on X

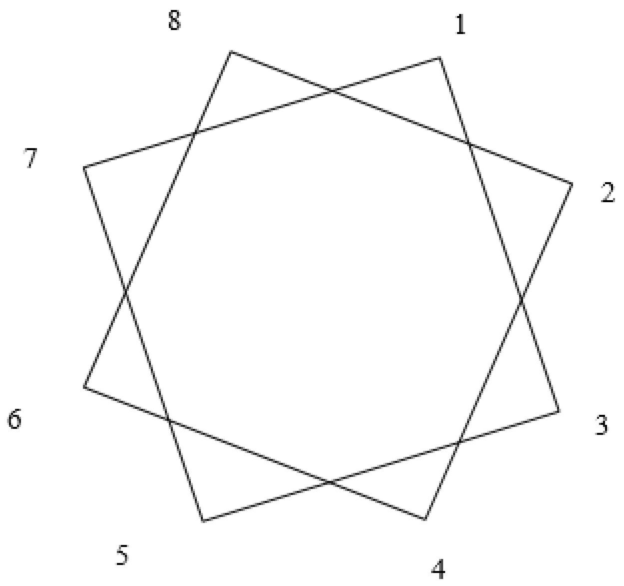

Figure 32: The graph Γ_2 of D_8 on X

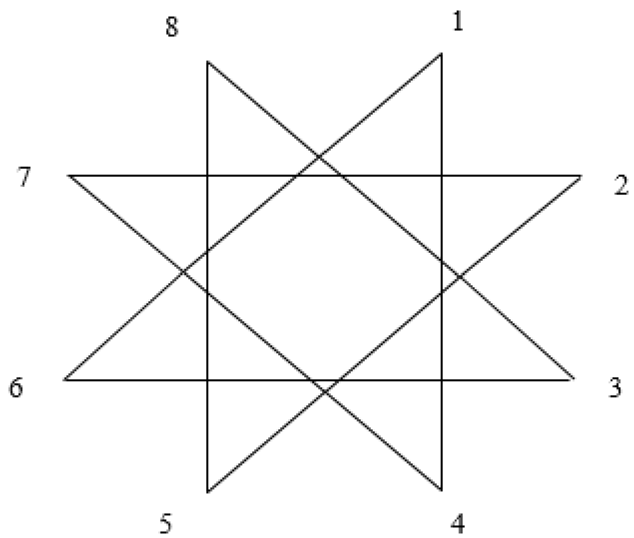

Figure 33: The graph Γ_3 of D_8 on X

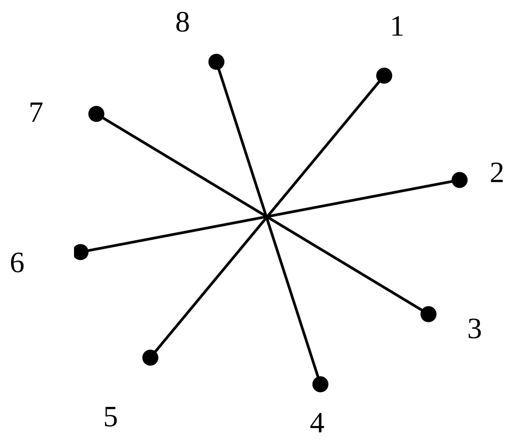

Figure 34: The graph Γ_4 of D_8 on X

4.4.4 Suborbital graphs of D_{10} on X

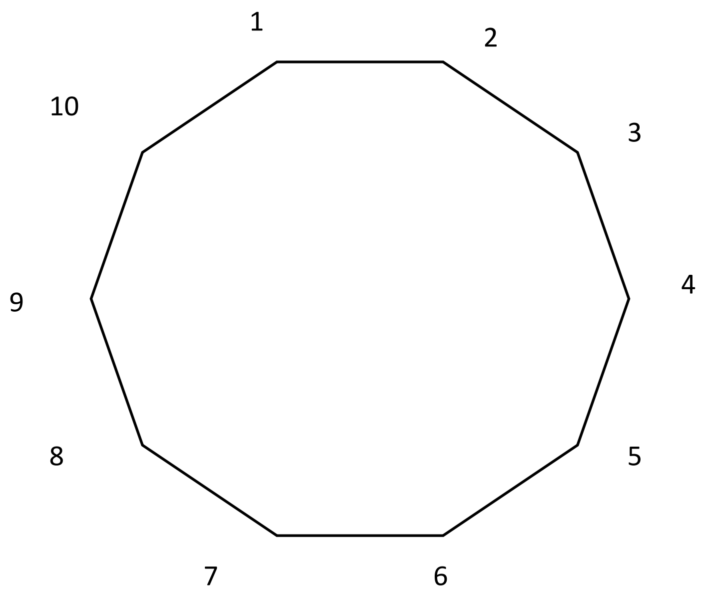

Figure 35: The graph Γ_1 of D_{10} on X

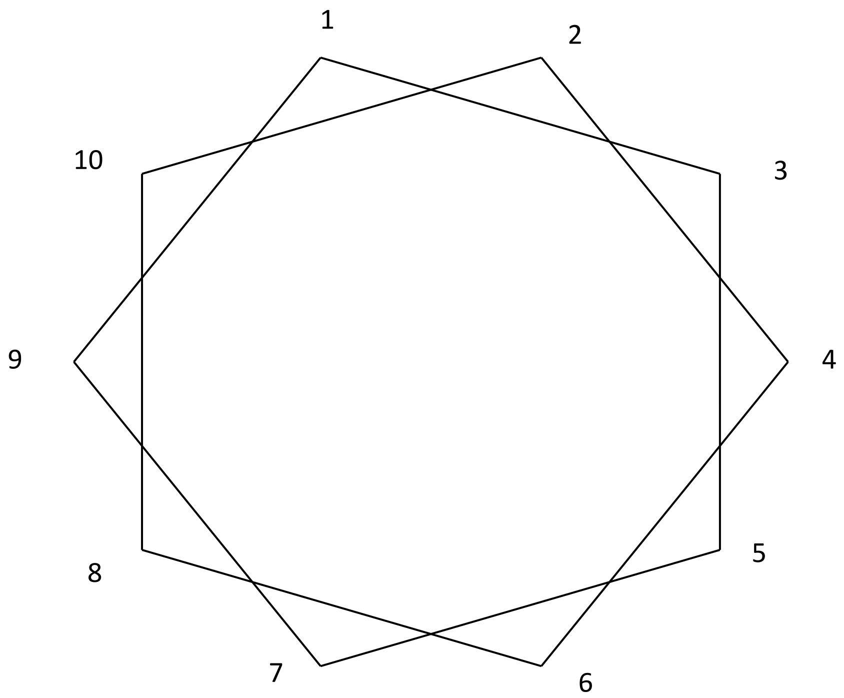

Figure 36: The graph Γ_2 of D_{10} on X

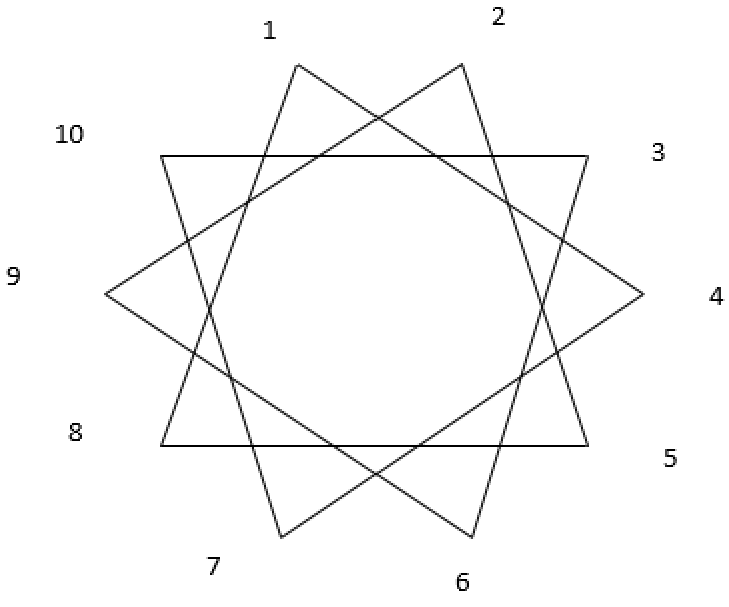

Figure 37: The graph Γ_3 of D_{10} on X

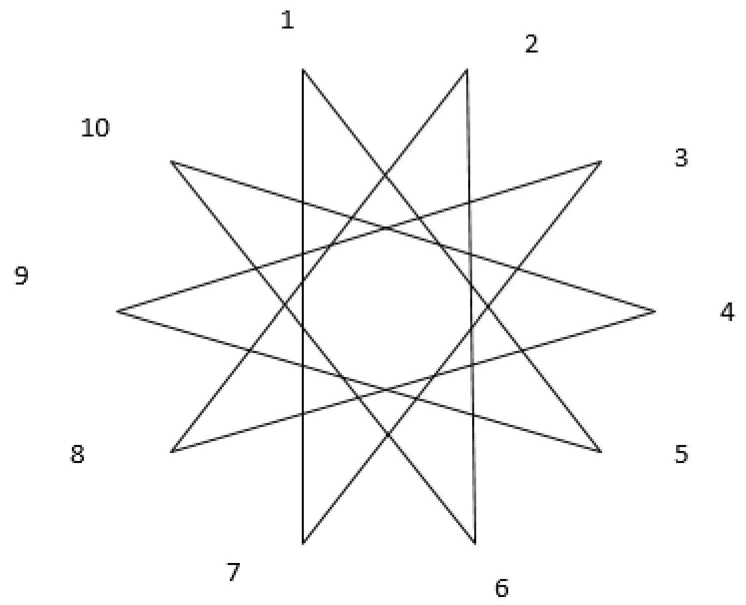

Figure 38: The graph Γ_4 of D_{10} on X

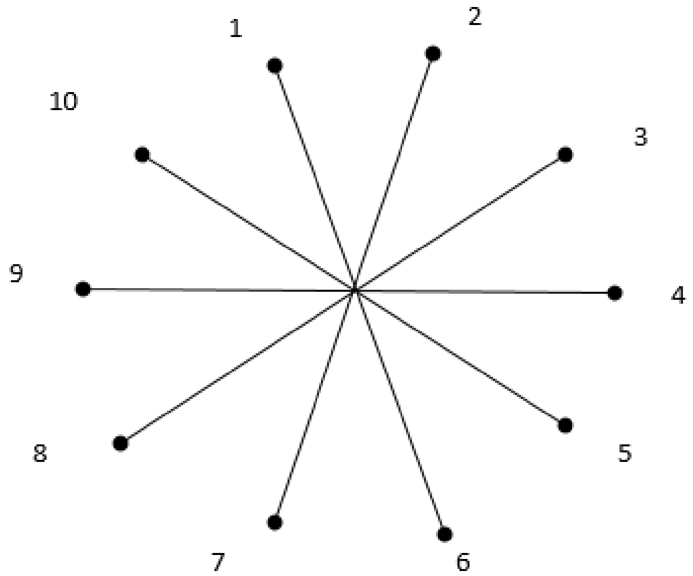

Figure 39: The graph Γ_5 of D_{10} on X

4.4.5 Properties of suborbital graphs of D_n acting on X

Theorem 4.4.2

All non-trivial suborbital graphs of D_n on X are undirected.

Proof:

Let Δ_i be the suborbit corresponding to the graph $\Gamma_{i.}$ From Corollary 4.1.16, Δ_i is self-paired. From Definition 1.6.15, $\Gamma_{i.}$ is undirected.

Theorem 4.4.3

The girth of the graph Γ_i of D_n on X is n/d, where d is gcd(i, n), provided $i \neq n/2$.

Proof:

Let Γ_i be a suborbital graph of D_n on X. From Theorem 4.4.1, the girth of Γ_i is the smallest integer k such that $1=1+ki \bmod n$, $2=2+ki \bmod n$, …, $l=l+ki \bmod n$. From the theory of cyclic groups, $k=n/d$, where d is gcd(i, n). If $i=n/2$, then Γ_i has a path joining exactly 2 vertices and the graph has no cycles.

Theorem 4.4.4

The graph Γ_i of D_n on X has d connected components, where d is gcd (i, n).

Proof:

From Theorem 4.4.3, there is a path in Γ_i from x to y if and only $x \equiv y \bmod d$. A complete residue system *modulo d* has d congruent classes, which are; 0, 1, 2, …, d-1.

Theorem 4.4.5

The graph Γ_i of D_n on X is connected if and only if i and n are coprime.

Proof:

From Theorem 4.4.4, Γ_i is connected if and only if d=1. This is possible only if i and n are coprime.

Theorem 4.4.6

The action of D_n on X is primitive if n is prime.

Proof:

Let Γ_i be a graph of D_n on X, where n is prime. Then, i and n are coprime, $1 \leq i \leq n$. From Theorem 4.4.5, the graph is connected. From Theorem 1.6.16, the action is primitive.

Theorem 4.4.7

The number of connected graphs Γ_i corresponding to the action of D_n on X is ½($\phi(n)$), where $\phi(n)$ is the totient phi function.

Proof:

From Theorem 4.4.5, Γ_i is connected if and only if i is relatively prime to n. The number of integers, less than or equal to n, that are relatively prime to n is $\phi(n)$. From Remark 4.1.18, Δ_i and Δ_{n-i} are the same suborbits. It follows, the number of connected graphs is ½($\phi(n)$).

Theorem 4.4.8

The graph Γ_i of D_n on X when gcd $(i, n)=d$, $d > 1$, has d connected components and each component has girth n/d.

Proof:

Let Γ_i be a graph of D_n on X, where gcd $(i, n)=d$, $d > 1$. From Theorems 4.4.3 and 4.4.4, Γ_i has d, n/d-cycles as shown in Figure 40.

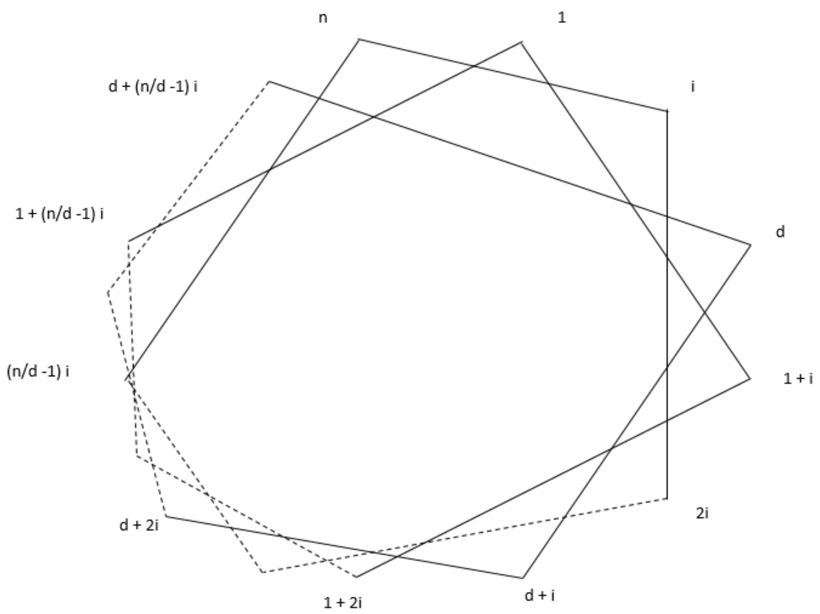

Figure 40: The graph гi of Dn on X when gcd (i, n)=d, d > 1

Theorem 4.4.9

The graph Γ_i has 1 connected component when i and n are coprime.

Proof:

From Theorems 4.4.3 and 4.4.4, Γ_i has d cycles, each of girth n/d. When i and n are coprime, d=1. It follows Γ_i has 1 cycle of length n as shown in Figure 41.

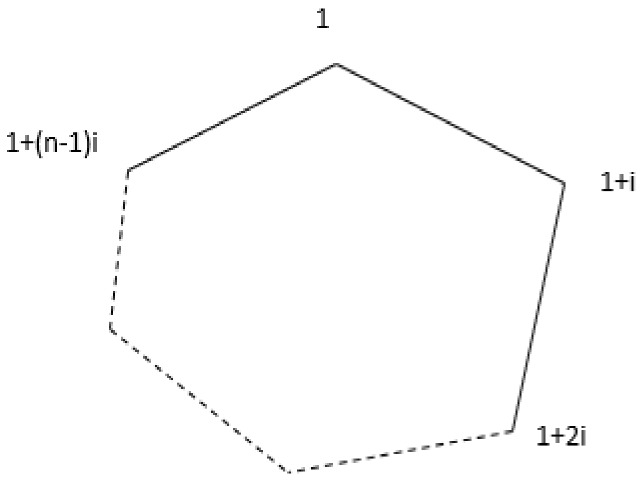

Figure 41: The graph Γ_i of D_n on X when i and n are coprime

Theorem 4.4.10

The graph Γ_i has no cycles when $i=n/2$.

Proof:

From Theorem 4.4.3, if $i=n/2$, then a path in Γ_i joins exactly 2 vertices. The graph is shown in Figure 42.

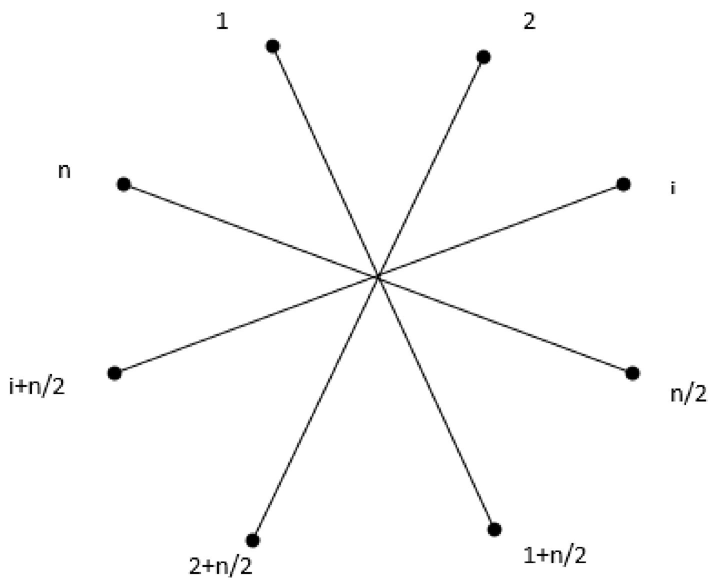

Figure 42: The graph Γ_i of D_n on X when $i = n/2$

4.5 Suborbitals and suborbital graphs of D_n acting on $X^{(n-1)}$

The suborbital O_i corresponding to suborbit Δ_i is the set $O_i=\{g(\Delta_0, A_i) \mid g \in G\}$, where $A_i \in \Delta_i=\{\{i+1, i+2, \ldots, i-1\}, \{1-i, 2-i, \ldots, n-1-i\}\}$. The suborbital graph Γ_i corresponding to Δ_i is constructed by considering $X^{(n-1)}$ as the vertices, and drawing an edge from C to D if and only if $(C, D) \in O_i$.

Theorem 4.5.1

Suppose O_i is the suborbital corresponding to Δ_i and $A_i=\{i+1, i+2, \ldots, i-1\} \in \Delta_i$. Then the pair $(C, D)=(\{c_1, c_2, \ldots, c_{n-1}\}, \{d_1, d_2, \ldots, d_{n-1}\}) \in O_i(\Delta_0, A_i)$ if and only if $d_k-c_k=i \bmod n$ or $d_k-c_k=n-i \bmod n$ $k=1, 2, \ldots, n-1$.

Proof:

If $(C, D) \in O_i(\Delta_0, A_i)$, then $[c_1, d_1]=hg^j\,[1, i+1] = [1+j, i+1+j]$ or $[i+1+j, 1+j]$, $\Longrightarrow d_1-c_1=i \bmod n$ or $n-i \bmod n$. Since the coordinates of C and D are consecutive integers $\bmod n$, $d_k-c_k=i \bmod n$ or $d_k-c_k=n-i \bmod n$, $k=1, 2, \ldots, n-1$. Conversely, if $d_k-c_k=i \bmod n$, then $d_k=c_k+i$ and $hg^{(ck-k)}(k, i+k)=h(c_k, d_k)=(d_k, c_k)$, $\Longrightarrow (C, D) \in O_i$ □

4.5.1 Suborbital graphs of D_6 acting on $X^{(5)}$

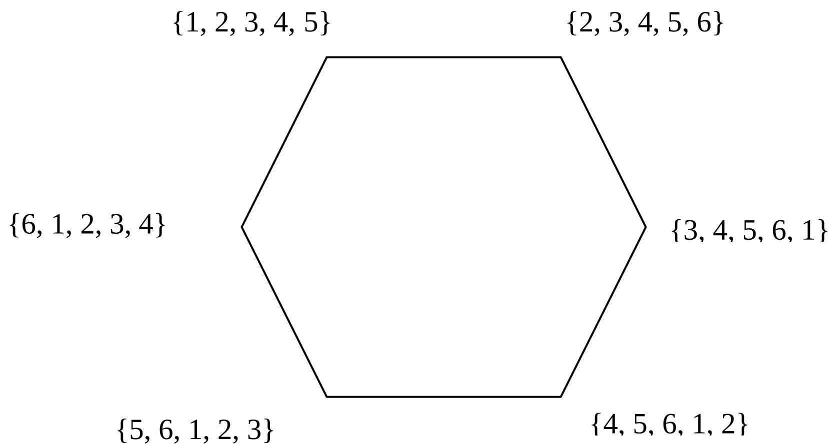

Figure 43: The graph Γ_1 of D_6 on $X^{(5)}$

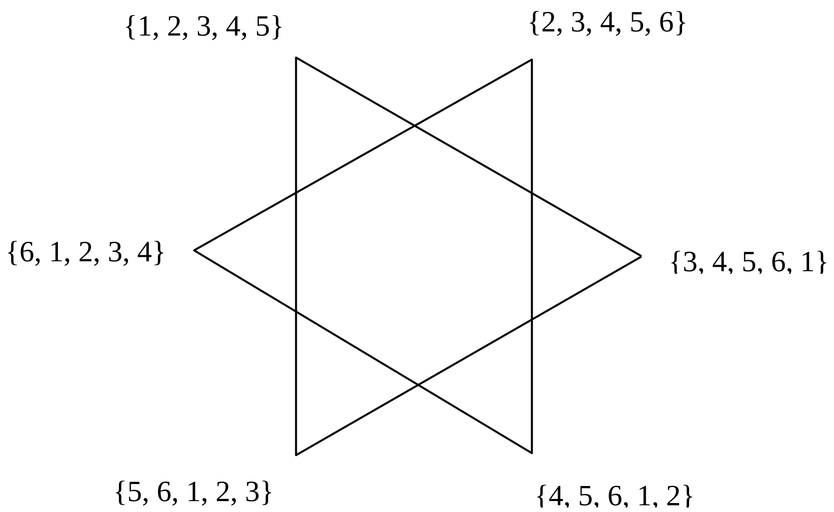

Figure 44: The graph Γ_2 of D_6 on $X^{(5)}$

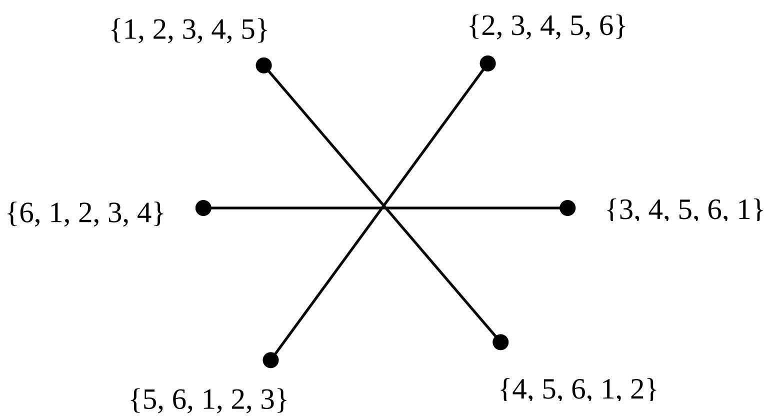

Figure 45: The graph Γ_3 of D_6 on $X^{(5)}$

4.5.2 Suborbital graphs of D_7 acting on $X^{(6)}$

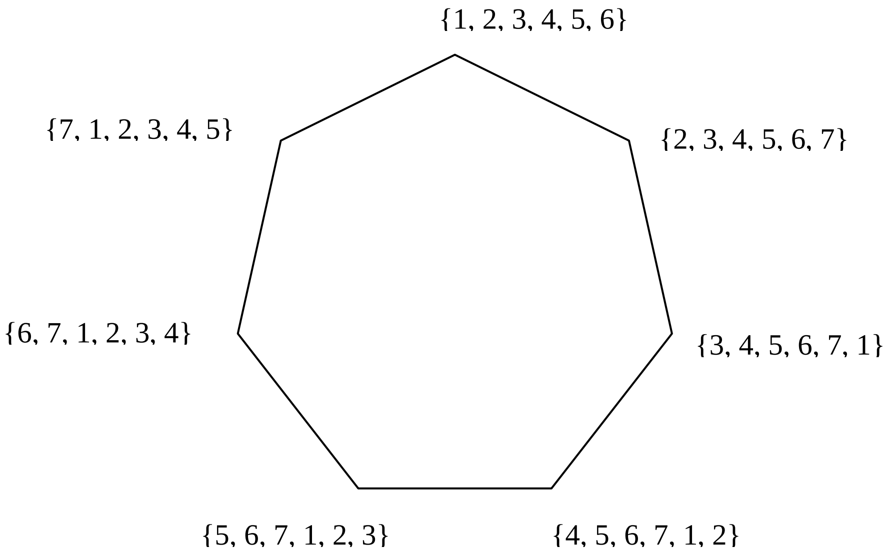

Figure 46: The graph Γ_1 of D_7 on $X^{(6)}$

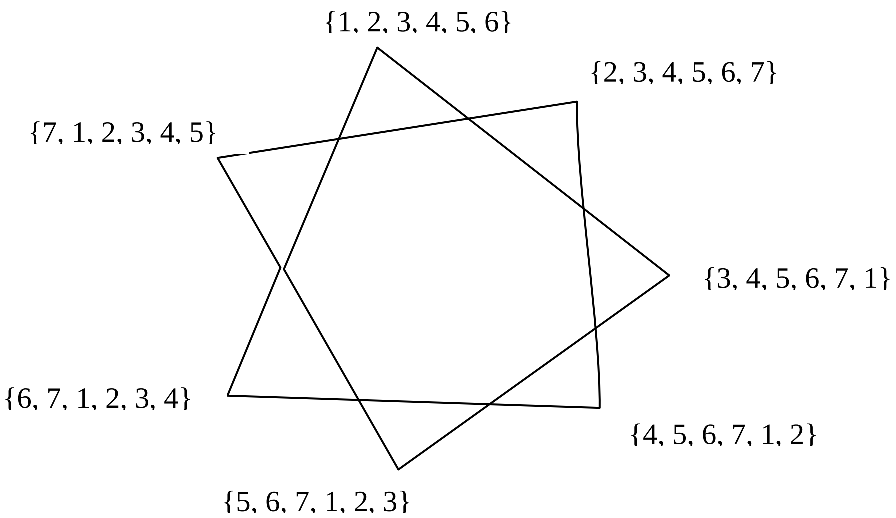

Figure 47: The graph Γ_2 of D_7 on $X^{(6)}$

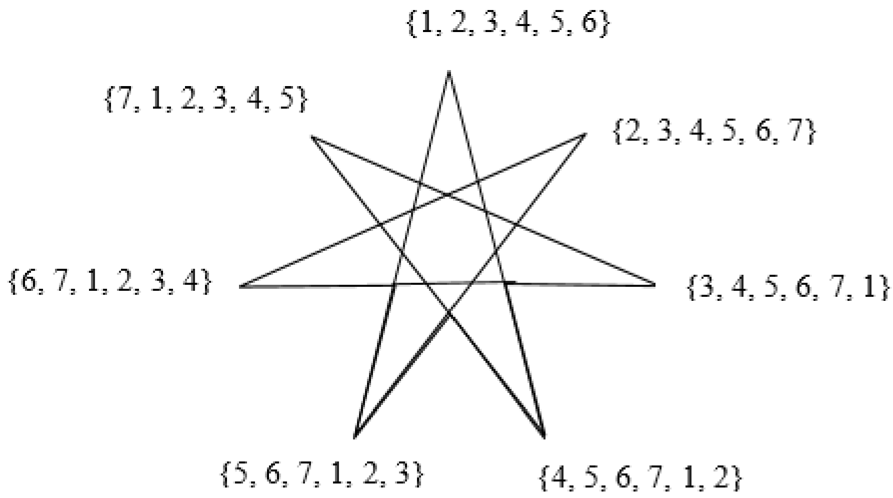

Figure 48: The graph Γ_3 of D_7 on $X^{(6)}$

4.5.3 Suborbital graphs of D_8 acting on $X^{(7)}$

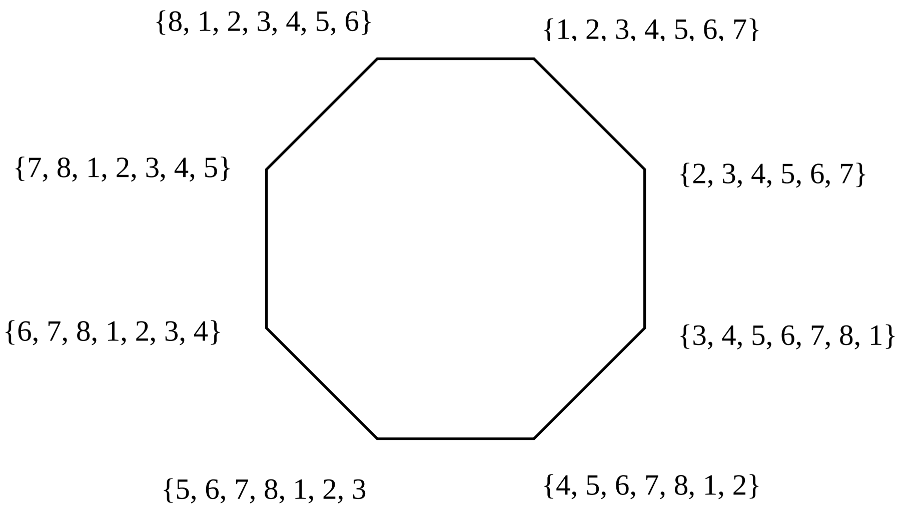

Figure 49: The graph Γ_1 of D_8 on $X^{(7)}$

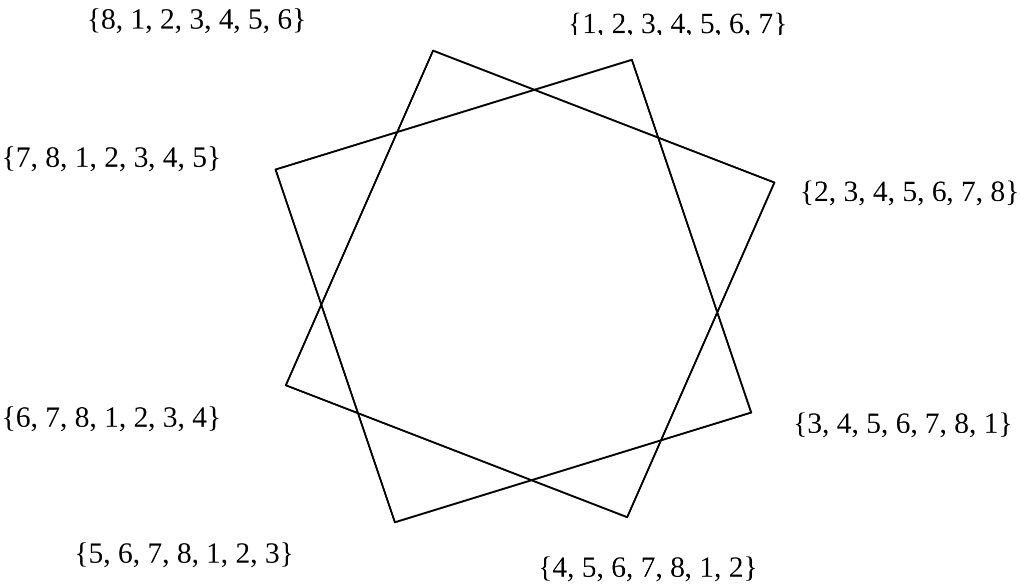

Figure 50: The graph Γ_2 of D_8 on $X^{(7)}$

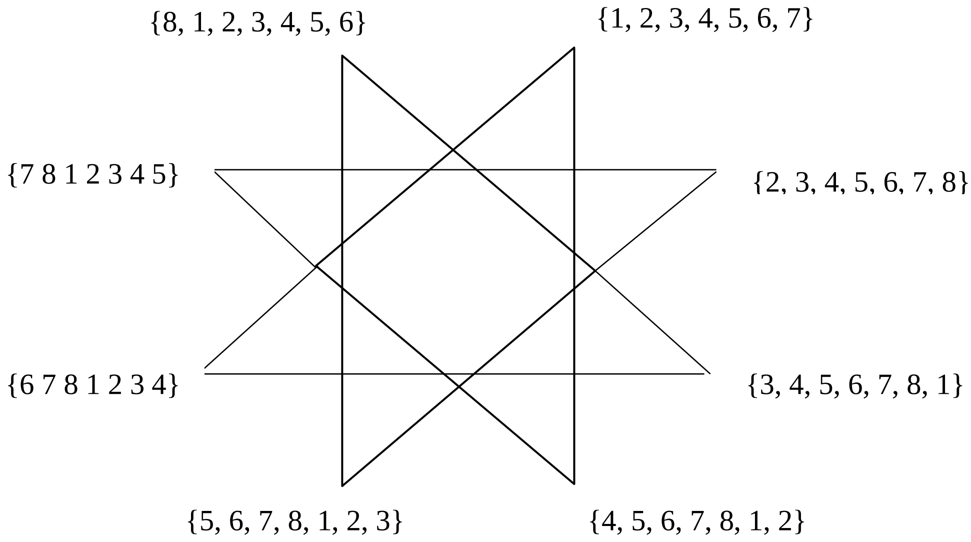

Figure 51: The graph Γ_3 of D_8 on $X^{(7)}$

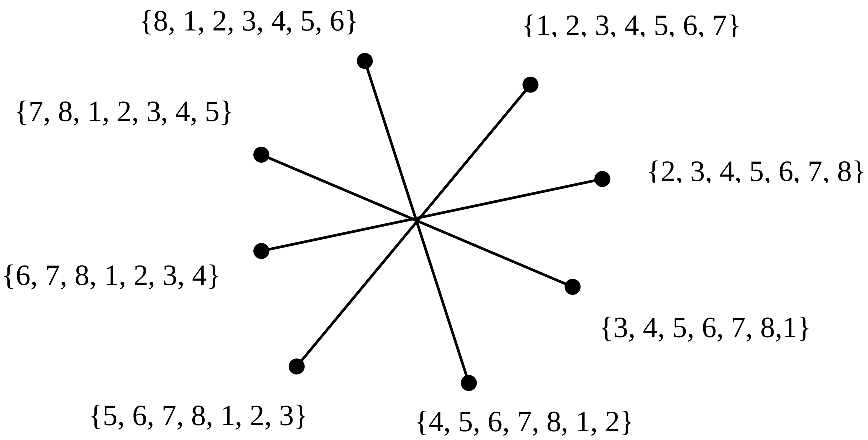

Figure 52: The graph Γ_4 of D_8 on $X^{(7)}$

Theorem 4.5.2

The action of D_n on X is equivalent to the action of D_n on $X^{(n-1)}$.

Proof:

Using Definition 1.6.21, let (G_1, X) and $(G_2, X^{(n-1)})$ be the action of D_n on X and the action of D_n on $X^{(n-1)}$, respectively. Define ϕ: $G_1 \to G_2$ such that $\phi(g)=g$, for all $g \in G_1$. Define Θ: $X \to X^{(n-1)}$ such that $\Theta(x)=X|x$, for all $x \in X$. Now, $\Theta(gx)= X|gx=g(X|x)= \phi(g)\, \Theta(x)$. □

Theorem 4.5.3

If i and n are coprime, then the graph Γ_i is connected. The graph is isomorphic to Γ_i shown in Figure 53 below.

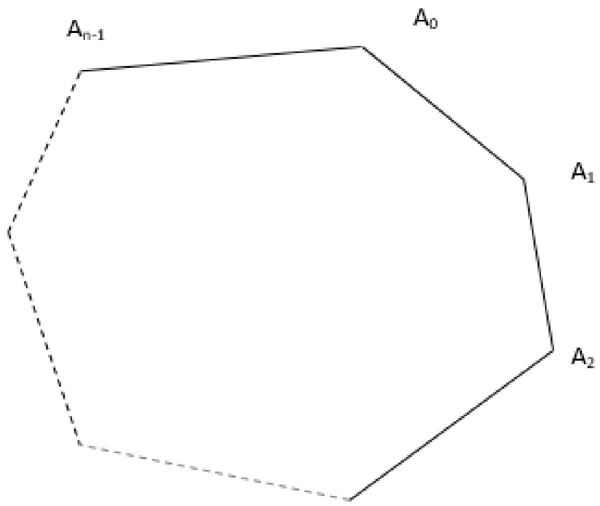

Figure 53: The graph Γ_i of D_n on $X^{(n-1)}$ when i and n are coprime

Proof:

From Theorem 4.5.2, the properties of suborbital graphs of D_n on X are the same as those of D_n on $X^{(n-1)}$. It follows, the graph takes the same form as Γ_i in Figure 41. Hence, the proof.

Theorem 4.5.4

If $i=n/2$, then the graph Γ_i is of the form shown in Figure 54 below.

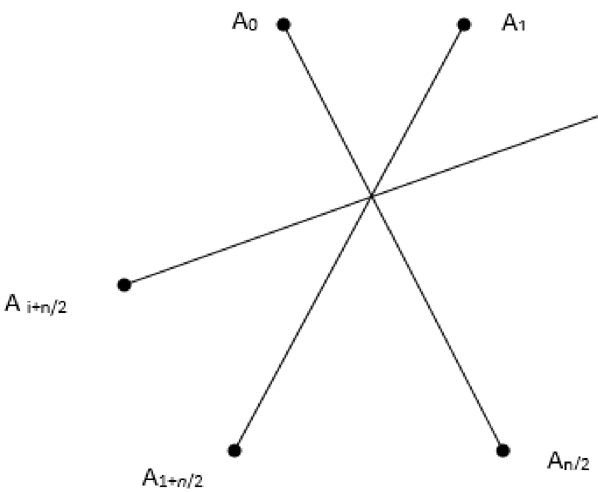

Figure 54: The graph Γ_i of D_n on $X^{(n-1)}$ when $i=n/2$

Proof:

From Theorem 4.5.2, Γ_i is of the same form as Γ_i in Figure 42. The graph appears as shown in Figure 54.

Theorem 4.5.5

If i and n have s common divisors in which gcd(i, n)=d, then the graph Γ_i has d connected components. The graph is shown in Figure 55 below.

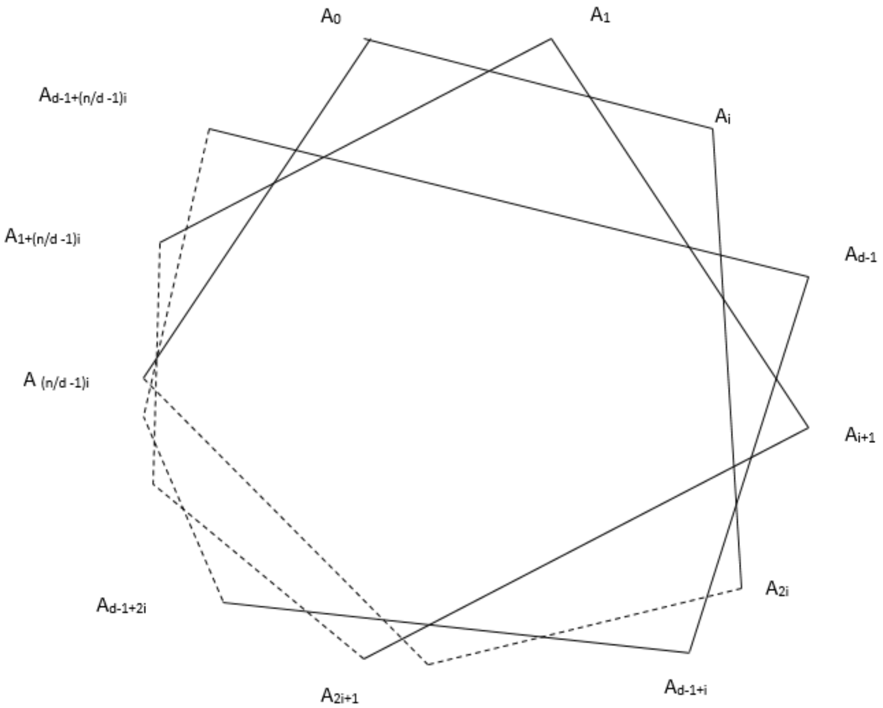

Figure 55: The graph Γ_i of D_n on $X^{(n-1)}$ when gcd(i, n)=d, $d > 1$

Proof:

From Theorem 4.5.2, and from Figure 40, Γ_i is of the form shown in Figure 55.

4.6 Action of C_n on $X^{[r]}$

The set $X^{[r]}$ comprises of all ordered r-element subsets of the set X={1, 2, …, n} and its cardinality is ${}^nP_r = \frac{n!}{(n-r)!}$. The action of a group G on $X^{[r]}$ is defined by; $g[x_1, x_2, \ldots, x_r]=[g(x_1), g(x_2), \ldots, g(x_r)]$, for all $[x_1, x_2, \ldots, x_r] \in X^{[r]}$, $g \in G$.

Theorem 4.6.1

The action of C_n on $X^{[r]}$ is transitive if and only if r=1 or n=r=2.

Proof:

Let $G=C_n$, $[x_1, x_2, ..., x_r]\in X^{[r]}$ and g in G. Now, g fixes any element $[x_1, x_2, \ldots, x_r]$ in $X^{[r]}$ if and only if $g[x_1, x_2, \ldots, x_r]=[x_1, x_2, \ldots, x_r]$ so that $g(x_1)=x_1$, $g(x_2)=x_2$, …, $g(x_r)=x_r$. This is possible if and only if g is the identity. The number of elements in $X^{[r]}$ fixed by the identity is ${}^{n}P_{r}$. Using Cauchy-Frobenius Lemma, the number of G-orbits on $X^{[r]}$ is;

$$\frac{1}{n}\left\{\frac{n!}{(n-r)!}\right\}=\frac{(n-1)!}{(n-r)!}$$

=1 if and only if r=1, $n\geq 1$ or $n=r=2$. But $r > 1$ for ordered elements.

Theorem 4.6.2

The rank of C_2 on $X^{[2]}$ is 2 and the subdegrees are; 1, 1.

Proof:

Let $G=C_2$ and $[1, 2]\in X^{[2]}$. Since $|G_{[1, 2]}|=1$, then $G_{[1, 2]}$ is the trivial subgroup of G. It follows that g in $G_{[1, 2]}$ fixes each element of $X^{[2]}$ in its own orbit. Hence, the rank of G on $X^{[2]}$ is 2 and the subdegrees are; 1, 1.The 2 suborbits of G are; Δ_0=[1, 2] and Δ_1=[2, 1], are self-paired.

4.6.1 Suborbital graph of C_2 on $X^{[2]}$

Let suborbital $O_1=\{g[[1, 2], [2, 1]]|g \in G\}$. Then the 2 elements of $X^{[2]}$ in this suborbital are; ([1, 2], [2, 1]). The corresponding suborbital graph is shown in Figure 56 below.

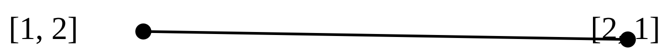

Figure 56: The graph Γ_i of C_2 on $X^{[2]}$

The graph is undirected with 1 connected component.

The action has only 1 non-trivial suborbit corresponding to 1 graph and thereby offering little to discuss.

4.7 Ranks, subdegrees and suborbital graphs of D_n on $X^{[r]}$

Theorem 4.7.1

The action of D_n on $X^{[r]}$ is transitive if and only if n=3 and $r\leq 3$.

Proof:

Let $g\in G=D_n$ and $[1, 2, \ldots, r]\in X^{[r]}$. Now, g fixes an ordered r-element subset if and only if g is the identity. Then $|G_{[1, 2, \ldots, r]}|=1$ and by Theorem 1.6.19, $|\text{orb}_G[1, 2, \ldots, r]|=|G|/1=2n$. For transitivity, $|X^{[r]}|=2n$, $\Rightarrow \left(\frac{n!}{(n-r)!}\right)=2n$, $\Rightarrow n=3$ and $r \leq 3$. Conversely, if $n=3$ and $r \leq 3$, then $|X^{[r]}|=2n$ and the action is transitive.

Theorem 4.7.2

The rank of $G=D_3$ on $X^{[r]}$ is 6 and each suborbit contains 1 element.

Proof:

Let the group $G_{[1, 2, \ldots, r]}$ act on $X^{[r]}$ and $[1, 2, \ldots, r] \in X^{[r]}$. From Theorem 4.7.1, g in $G_{[1, 2, \ldots, r]}$ fixes each element of $X^{[r]}$ in its own $G_{[1, 2, \ldots, r]}$- orbit. Since $|X^{[r]}|=6$ then the rank of G on $X^{[r]}$ is 6.

Clearly, the subdegrees of G are; 1, 1, 1, 1, 1, 1.

The 6 suborbits of G on $X^{[3]}$ are;

$\Delta_0=\{[1, 2, 3]\}$, $\Delta_1=\{[1, 3, 2]\}$, $\Delta_2=\{[3, 2, 1]\}$, $\Delta_3=\{[2, 1, 3]\}$, $\Delta_4=\{[2, 3, 1]\}$, $\Delta_5=\{[3, 1, 2]\}$.

Theorem 4.7.3

The number of self- paired suborbits of D_3 on $X^{[3]}$ is 4.

Proof:

Let $G=D_3$ act on $X^{[3]}$. Now, g^2 fixes an ordered set of r elements if g is the identity or g is a reflection. If g is the identity, then $|Fix(g^2)|={}^3P_r=6$. If g is a reflection, then $|Fix(g^2)|={}^3P_r=6$. Using Theorem 1.6.17, the number of self-paired suborbits is $\frac{1}{|D_3|}\sum_{g\in D_3}|Fix(g^2)|=\frac{1}{6}\{6+3(6)\}=4$. □

The 4 self- paired suborbits of D_3 on $X^{[2]}$ are; $\Delta_0=[1, 2]$, $\Delta_1=[1, 3]$, $\Delta_2=[3, 2]$ and $\Delta_3=[2, 1]$. By definition of sef-pairedness, $g[\Delta_0, \Delta_1]=[\Delta_1, \Delta_0]$ when $g=(23)$, $g[\Delta_0, \Delta_2]=[\Delta_2, \Delta_0]$ when $g=(13)$ and $g[\Delta_0, \Delta_3]=[\Delta_3, \Delta_0]$ when $g=(12)$.

The 4 self- paired suborbits of D_3 on $X^{[3]}$ are; $\Delta_0=[1, 2, 3]$, $\Delta_1=[1, 3, 2]$, $\Delta_2=[3, 2, 1]$, $\Delta_3=[2, 1, 3]$. Clearly, $g[\Delta_0, \Delta_1]=[\Delta_1, \Delta_0]$ when $g=(23)$, $g[\Delta_0, \Delta_2]=[\Delta_2, \Delta_0]$ when $g=(13)$ and $g[\Delta_0, \Delta_3]=[\Delta_3, \Delta_0]$ when $g=(12)$.

4.7.1 Suborbitals and suborbital graphs of $G=D_3$ acting on $X^{[r]}$

Let $\Delta=[x_1, \ldots, x_r]$ be a suborbit of G on $X^{[r]}$, where $x_i \in \{1, 2, \ldots, n\}$. Then the suborbital O corresponding to Δ is given by; $O=\{(g[1, 2, \ldots, r], g[x_1, x_2, \ldots, x_r])|g \in G, [x_1, x_2, \ldots, x_r] \in \Delta\}$. The graph г corresponding to suborbital O is formed by considering $X^{[r]}$ as the vertex set and drawing an edge from $[c_1, c_2, \ldots, c_r]$ to $[d_1, d_2, \ldots, d_r]$ if and only if $([c_1, c_2, \ldots, c_r], [d_1, d_2, \ldots, d_r]) \in O$. Now, the suborbital graph corresponding to a self-paired suborbit Δ_i has an edge from C=[c_1, c_2, …, c_r] to D=[d_1, d_2, …, d_r] only if $|C \cap D|$=1. The graph corresponding to a paired suborbit Δ_i has an edge from C=[c_1, c_2, …, c_r] to D=[d_1, d_2, …, d_r] only if $|C \cap D|$=0.

Theorem 4.7.4

Let O_i be the suborbital corresponding to a self-paired suborbit Δ_i of D_3. Then $(C, D) \in O_i(\Delta_0, \Delta_i)$ if and only if the k^{th} coordinate of C is identical to the k^{th} coordinate of D.

Proof:

If $(C, D) \in O_i(\Delta_0, \Delta_i)$, then $(C, D)=g(\Delta_0, \Delta_i)= (\Delta_i, \Delta_0)$. Now, D=[1, 2, 3] and $C=g\Delta_0$, where g fixes either 1, or 2 or 3 in D. It follows, the k^{th} coordinate of C is identical to the k^{th} coordinate of D. Conversely, if the k^{th} coordinate of C is identical to the k^{th} coordinate of D, then $g(C, D)=(D, C)$ and therefore, $(C, D) \in O_i(\Delta_0, \Delta_i)$.

Theorem 4.7.5

Let O_i be the suborbital corresponding to a paired suborbit Δ_i of D_3. Then $(C, D) \in O_i(\Delta_0, \Delta_i)$, if and only if and the k^{th} coordinate of D is j *mod* 3 more than the k^{th} coordinate of C , where $g^j\Delta_0=\Delta_i$.

Proof:

Suppose Δ_i is a paired suborbit of G and $(C, D) \in O_i(\Delta_0, \Delta_i)$. Then $(C, D) =g^j(\Delta_0, \Delta_i)$, and $(D, C) =g^{-j}(\Delta_0, \Delta_i)$. Now, C=[1+j, 2+j, 3+j] and D=[1-j, 2-j, 3-j]. Clearly, the coordinate at D differs with the one at C by j *mod* 3. Conversely, if the coordinates in (C, D), differ by j, then $(C, D) =g^j(\Delta_0, \Delta_i)$ and therefore $(C, D) \in O_i(\Delta_0, \Delta_i)$.

4.7.2 Suborbital graphs of D_3 on $X^{[2]}$

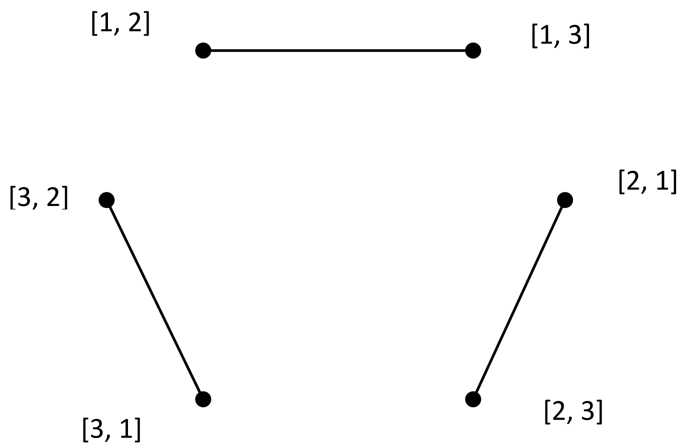

Figure 57: The graph Γ_1 of D_3 on $X^{[2]}$

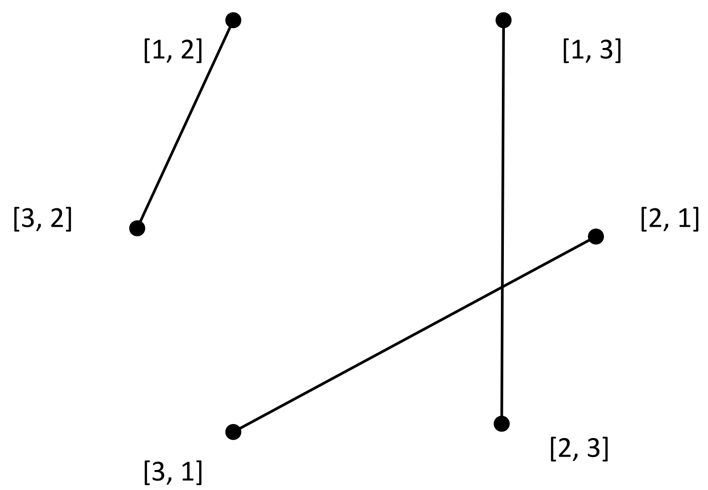

Figure 58: The graph Γ_2 of D_3 on $X^{[2]}$

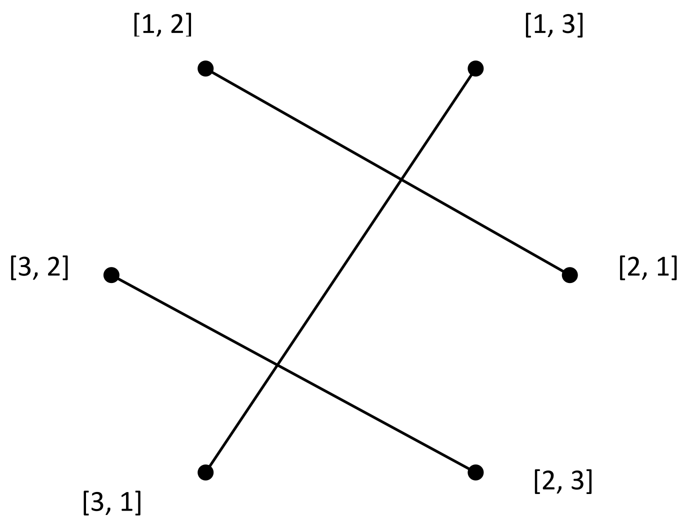

Figure 59: The graph Γ_3 of D_3 on $X^{[2]}$

[1, 2]
[1, 3]

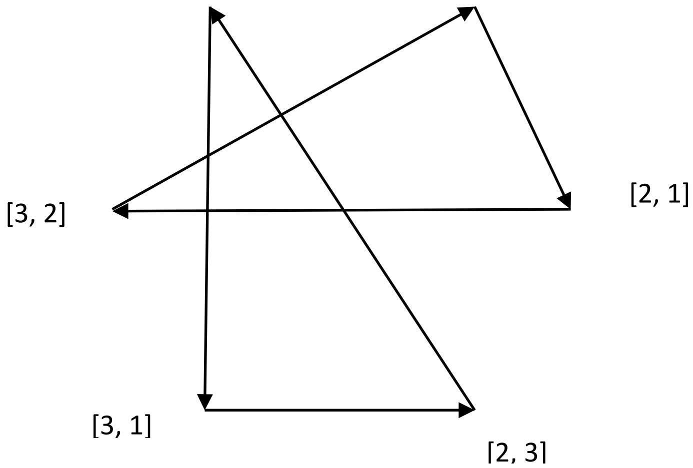

Figure 60: The graph Γ_4 of D_3 on $X^{[2]}$

The graph Γ_5 is paired with Γ_4 as the respective suborbits are paired.

4.7.3 Suborbital graphs of D_3 on $X^{[3]}$

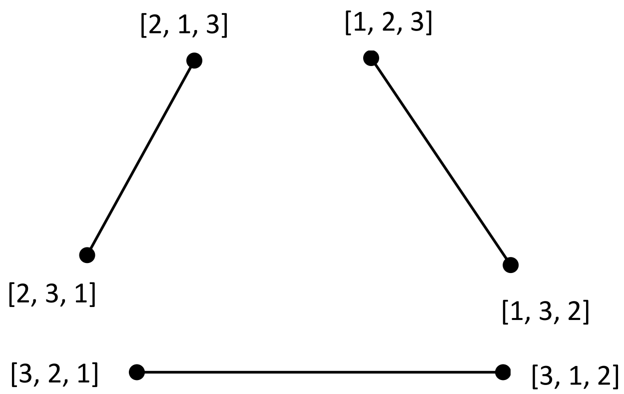

Figure 61: The graph Γ_1 of D_3 on $X^{[3]}$

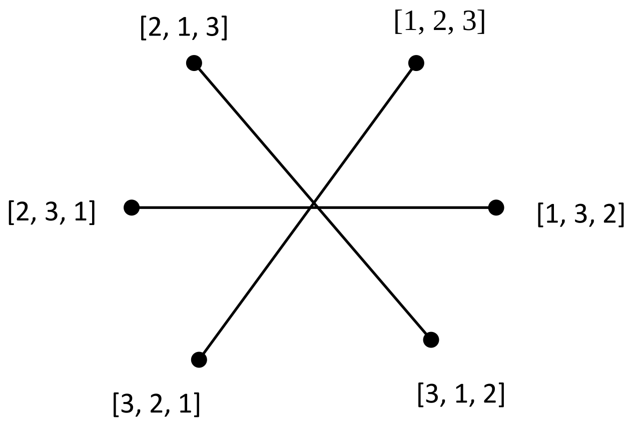

Figure 62: The graph Γ_2 of D_3 on $X^{[3]}$

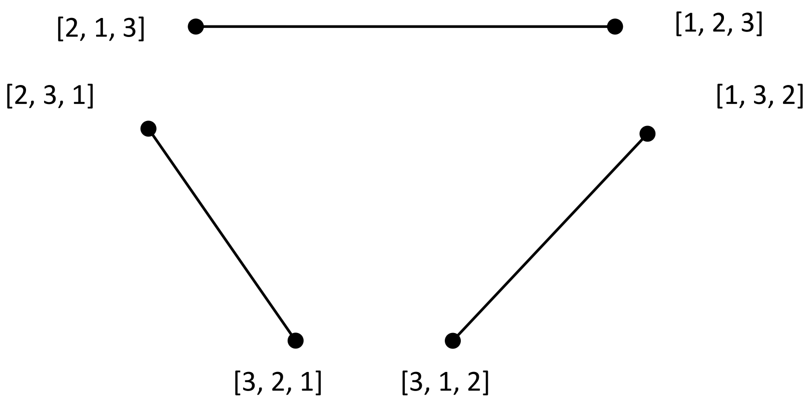

Figure 63: The graph Γ_3 of D_3 on $X^{[3]}$

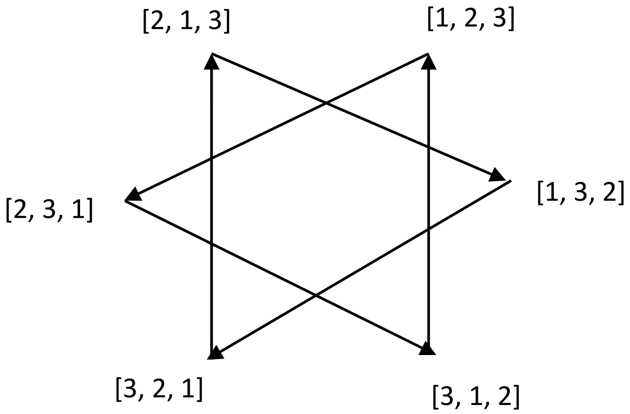

Figure 64: The graph Γ_4 of D_3 on $X^{[3]}$

The graph Γ_5 is paired with Γ_4.

4.7.4 Properties of suborbital graphs of D_3 on $X^{[r]}$

Theorem 4.7.6

All suborbital graphs of D_3 on $X^{[r]}$ are disconnected.

Proof:

Clearly, all the graphs have been constructed and they are all disconnected.

Theorem 4.7.7

The action of D_3 on $X^{[r]}$ is imprimitive.

Proof:

Suppose Γ_i corresponds to suborbit, Δ_i, of D_3 on $X^{[3]}$. Then, from Theorem 4.7.6, the graph is disconnected. From Theorem 1.6.16, the action of D_3 on $X^{[r]}$ is imprimitive.

Theorem 4.7.8

The number of connected components of Γ_i corresponding to a self-paired Δ_i is 3, while the number of those corresponding to a paired Δ_i is 2.

Proof:

Clearly, the proof follows from the graphs and corresponding suborbits by exhaustive method.

4.8 Rank, suborbits and suborbital graphs of $H=\left\langle\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\right\rangle$ on $\mathbb{Z}_\mathbf{p}$

4.8.1 Action of $H=\left\langle\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\right\rangle$ on $\mathbb{Z}_\mathbf{p}$

The group H has elements of the form

$$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} | h \in \mathbb{Z}_\mathbf{p}.$$

The action of H on $X=\mathbb{Z}_\mathbf{p}$ is defined by

$$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} : \begin{pmatrix}x\\ 1\end{pmatrix} \to \begin{pmatrix}x+h\\ 1\end{pmatrix}, \text{ for all } x \in X.$$

Theorem 4.8.1

The action of H on $\mathbb{Z}_\mathbf{p}$ is transitive.

Proof:

Let H act on $X=\mathbb{Z}_\mathbf{p}$, $x\in X$, and $\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\in H$. The stabilizer of x in H is the identity in H, since

$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix}=\begin{pmatrix}x+h\\ 1\end{pmatrix}$, $\Rightarrow$ $\{x+h \ | h\in X\}=\{x\}$ only if $h=0$. By Theorem 1.6.19, $|orb_H(x)|=|H|/1=p=|X|$ and the action is transitive.

Theorem 4.8.2

The rank of H on X is p and the length of each suborbit is 1.

Proof:

Let H_0 act on X. From Theorem 4.8.1 H_0, the identity in H_0 fixes each element of X in its own orbit. The number of H_0-orbits on X is $p/1=p$, the rank of H on X. Clearly, the length of each suborbit is 1 and the subdegrees are; 1, 1, …, 1 (p ones). The p suborbits of H are;

$\Delta_0=\{0\}$, $\Delta_1=\{1\}$, …, Δ_{p-1}, where $\Delta_i=\{i\}$.

Theorem 4.8.3

Let H act on X. Then Δ_i and Δ_j are paired if and only if $i+j\equiv 0 \ mod\ p$.

Proof:

Suppose Δ_i and Δ_j are paired suborbits of H. Then there exists h in X such that $i+h=0 \ mod\ p$ and $h=j \ mod\ p$, from the definition of pairedness. It follows, $i+j\equiv 0 \ mod\ p$. Conversely, if $i+j\equiv 0 \ mod\ p$, then Δ_i and Δ_j are paired when $h=j \ mod\ p$.

Corollary 4.8.4

Let H act on X. Then Δ_i is self- paired if and only if $i=0 \ mod\ p$.

Proof:

From Theorem 4.8.3, Δ_i is self- paired if and only if $i=j$ in the equation, $i+j\equiv 0 \ mod\ p$. It follows, $2i=0 \ mod\ p$, $\Rightarrow i=0 \ mod\ p$.

Corollary 4.8.5

All the non-trivial suborbits of H on X are paired.

Proof:

From Corollary 4.8.4, Δ_i is self- paired only if $i=0$. It follows, all non-trivial suborbits are paired.

Example 4.8.5

Suppose $X=\mathbb{Z}_{11}=\{0, 1, \ldots, 10\}$. If $x \in X$ and $H=\{\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix} \mid h \in X\}$, then the stabilizer of x in H is the identity in H, $H_x=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. The 11 suborbits of H on X are; $\Delta_0=\{0\}$, $\Delta_1=\{1\}$, …, $\Delta_{10}=\{10\}$. To obtain Δ_i^*, solve for j in $j+i=0 \ mod\ p$. It follows, $j=p-i \ mod\ p$. Hence, $\Delta_i^*=\Delta_{p-i}$.

4.8.2 Suborbitals and suborbital graphs of *H* acting on $X=\mathbb{Z}_p$

Let H act on X. The suborbital corresponding to Δ_i is the set $O_i=\{(h, i+h) \mid h \in X$. The corresponding suborbital graph Γ_i has an edge from x to y if and only if $(x, y) \in O_i$.

Theorem 4.8.6

For any two elements, x and y, in X, the ordered pair $(x, y) \in O_i$ if and only if $y\text{-}x \equiv i \mod p$.

Proof:

Let $H=\{\begin{pmatrix}1 & h\\0 & 1\end{pmatrix} | \ h \in X)\}$ act on X, and suppose Δ_i corresponds to O_i. If (x, y) is in O_i, then $(x, y) = (h, i+h) \mod p$. It follows, $y\text{-}\ x \equiv i \mod p$. Conversely, if $y\text{-}\ x \equiv i \mod p$, then $y \equiv x + i \mod p$ and $(x, x + i)$ is in O_i. □

Theorem 4.8.7

Let H act on X. The graph Γ_i is paired with Γ_j if and only if $i+j \equiv 0 \mod p$.

Proof:

Suppose Δ_i and Δ_j are paired suborbits of H, corresponding to Γ_i and Γ_j respectively. Then the proof follows from Theorem 4.8.3.

Theorem 4.8.8

The graph Γ_i is undirected if and only if $i \equiv 0 \mod p$.

Proof:

From Theorem 4.8.7, Γ_i is undirected if $i=j$ in the equation if $i+j \equiv 0 \mod p$. Hence, $i \equiv 0 \mod p$.

Theorem 4.8.9

The girth of Γ_i is p for all $i > 0$.

Proof:

From Theorem 4.8.6, Γ_i has edges of the form, $0 \rightarrow i \rightarrow 2i \rightarrow \ldots \rightarrow ki$, $1 \rightarrow 1+i \rightarrow 1+2i \rightarrow \ldots 1+mi$, …

The length of the shortest cycle of Γ_i is the smallest positive integer k such that $ki=0 \mod p$. It follows, $k=p$.

Theorem 4.8.10

The graph Γ_i is connected for all $i > 0$.

Proof:

From Theorem 4.8.9, there is a path joining any two vertices of Γ_i. The graph is connected from Definition 1.6.12.

Theorem 4.8.11

The action of H on X is primitive.

Proof:

From Theorem 4.8.10, Γ_i is connected. The action is primitive by Theorem1.6.16.

Theorem 4.8.12

The graph Γ_i is isomorphic to the graph shown in Figure 65.

Proof:

From Theorem 4.8.9 and Theorem 4.8,10, Γ_i is of the form shown in Figure 65.

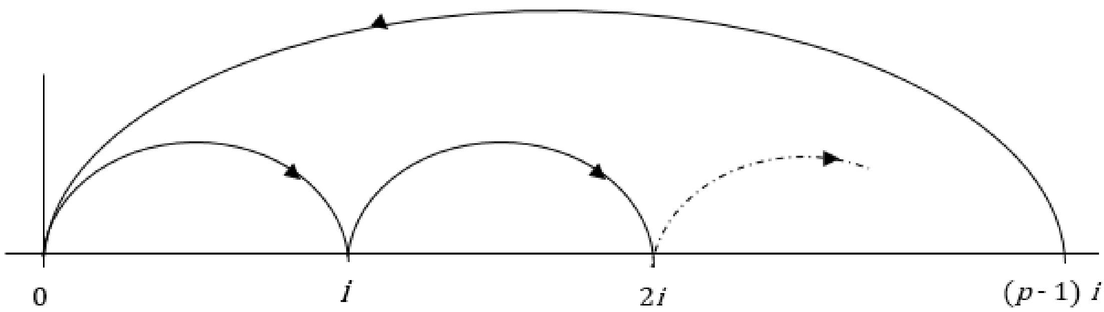

Figure 65*:* The graph Γ_i of *H* on $\mathbb{Z}_p$

CHAPTER FIVE
CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The study intended to compute the ranks, subdegrees and suborbital graphs of the actions of C_n, D_n , and H on finite sets. This was achieved, in line with the stated objectives, by using applicable theory in the area.

Transitivity was determined where each of the actions of D_n and C_n on $X^{(r)}$ is transitive if and only if r=1, r=n-1 or r=n. However, the action of D_n on $X^{[r]}$ is transitive only if n=3 and $r \leq 3$, whereas that of C_n on $X^{[r]}$ is transitive if and only if r=1 or n=r=2. The action of $H=\langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\rangle$ on $\mathbb{Z}_\mathbf{p}$ has also been found to be transitive.

The rank of C_n on X was proved to be n, and equal to the rank of C_n on $X^{(n-1)}$. Consequently, the subdegrees of each of the actions have been shown to be; 1, 1, …, 1 (n ones). Similarly, it has been proved that the rank of D_n on X is equal to the rank of D_n on $X^{(n-1)}$ and each of them is (n+1)/2 when n is odd, and (n+2)/2 when n is even. In each of the actions of D_n, there was 1 suborbit of length 1 and (n-1)/2 suborbits of length 2 when n is odd. But there were 2 suborbits of length 1 and (n-2)/2 suborbits of length 2 when n is even. On the other hand, the rank of D_3 on $X^{[3]}$ and on $X^{[2]}$ was found to be 6, while that of C_2 on $X^{[2]}$ was 2.The rank of H on $\mathbb{Z}_\mathbf{p}$ has been shown to be p and the subdegrees are; 1, 1, …, 1(p ones).

All non-trivial suborbits of the action of D_n, on X were self-paired. But only 1 non-trivial suborbit of the action of C_n on X was self-paired, when n is even. The action of H on $\mathbb{Z}_\mathbf{p}$ has no non-trivial self-paired suborbits. These results are verified by Theorem 1.6.16.
From each of the actions of D_n and C_n on X, the graph Γ_i had d, n/d cycles, where d=gcd(i, n). This resulted to 3 graphs, one of which was connected and the other 2 were disconnected. From the 2 disconnected, 1 of them had cycles and the other had no cycles (see Figures 40, 41 and 42 and Theorem 4.3.5 respectively). The graphs of D_n and C_n on $X^{(n-1)}$ followed the same generalization of the 3 graphs ((see Figures 53, 54 and 55) and (Figures 22, 23 and 24) respectively). However, the graph of H on $\mathbb{Z}_\mathbf{p}$ was only 1 which is connected (see Figure 65).

From the actions of D_n and C_n, it has been established that the corresponding graph is connected only if n is prime. This result implies that the respective groups are primitive only if n is prime.

5.2 Recommendations, application and further research

The action of G on X induces the action of G on $X^{(n\text{-}1)}$ from each of the groups, D_n and C_n. The equivalence exhibited by the action of G on X and that of G on $X^{(n\text{-}1)}$, offers a facility to study the action of G on $X^{(n\text{-}1)}$ through the action of G on X. The result is also applicable in Category theory in mapping of sets to elements. The results on primitive actions can be used to investigate primitivity of the groups on various sets of cardinality p. Further research could be done on the action of the groups on diagonals of a regular polygon.

The study conjectures that the action of $H=<\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}>$ on $\mathbb{Z}_\mathbf{p}$ is equivalent to that of C_n on X if $n=p$.

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